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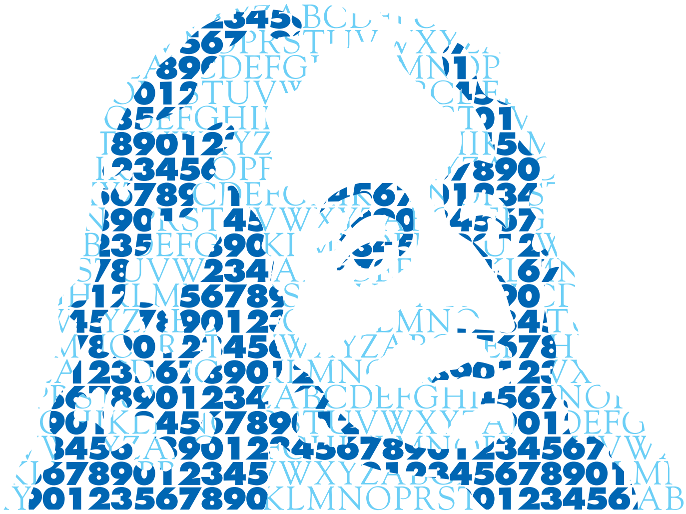

BLAISE PASCAL

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Volume 18, nº 2 (2011), p. 251-271.

<http://ambp.cedram.org/item?id=AMBP_2011__18_2_251_0>

Publication éditée par le laboratoire de mathématiques
de l'université Blaise-Pascal, UMR 6620 du CNRS
Clermont-Ferrand — France

cedram

Article mis en ligne dans le cadre du
Centre de diffusion des revues académiques de mathématiques
http://www.cedram.org/

Annales mathématiques Blaise Pascal **18**, 251-271 (2011)

Quantitative Isoperimetric Inequalities on the Real Line

Yohann de Castro

Abstract

In a recent paper A. Cianchi, N. Fusco, F. Maggi, and A. Pratelli have shown that, in the Gauss space, a set of given measure and almost minimal Gauss boundary measure is necessarily close to be a half-space.

Using only geometric tools, we extend their result to all symmetric log-concave measures on the real line. We give sharp quantitative isoperimetric inequalities and prove that among sets of given measure and given asymmetry (distance to half line, i.e. distance to sets of minimal perimeter), the intervals or complements of intervals have minimal perimeter.

Inégalités Isopérimétriques Quantitatives sur la Droite Réelle

Résumé

Dans un récent papier, A. Cianchi, N. Fusco, F. Maggi, et A. Pratelli ont montré que, dans l'espace de Gauss, un ensemble de mesure donnée et de frontière de Gauss presque minimal est nécessairement proche d'être un demi-espace.

En utilisant uniquement des outils géométriques, nous étendons leur résultat au cas des mesures log-concaves symétriques sur la droite réelle. On donne des inegalités isopérimétriques quantitatives optimales et l'on prouve que parmi les ensembles de mesure donnée et d'asymétrie donnée (distance à la demi-droite, i.e. distance aux ensembles de périmètre minimal), les intervalles ou les complémentaires d'intervalles ont le plus petit périmètre.

In this paper, we study the Gaussian isoperimetric inequality in dimension $n = 1$ and we prove a sharp quantitative version of it. More precisely, denote the one-dimensional Gaussian measure by

$$\gamma := \exp(-t^2/2)/\sqrt{2\pi} \cdot \mathcal{L}^1,$$

where $\mathcal{L}^1$ is the one-dimensional Lebesgue measure. The classical Gaussian isoperimetric inequality [6] states that among sets of given measure in

Keywords: Isoperimetric inequalities, Asymmetry, Log-concave measures, Gaussian measure.
Math. classification: 26B15, 49Q15.

$(\mathbb{R}^n, \gamma^n)$, where γ^n denotes the standard n-dimensional Gaussian measure, half spaces have minimal Gauss boundary measure. This reads as

$$P_{\gamma^n}(\Omega) \geq J_\gamma\left(\gamma^n(\Omega)\right),$$

where J_γ is optimal (and defined later on in the text). In their paper [4] A. Cianchi, N. Fusco, F. Maggi, and A. Pratelli have derived an improvement of the form

$$P_{\gamma^n}(\Omega) - J_\gamma(\gamma^n(\Omega)) \geq \Theta_{\gamma^n}\left(\gamma^n(\Omega), \lambda(\Omega)\right) \geq 0,$$

where $\lambda(\Omega)$ measures (in a suitable sense, see formula (1.6) below) how far Ω is from a half-space, and Θ_{γ^n} is a function of two variables, whose form depends on the reference measure γ^n, and such that $\Theta_{\gamma^n}(x, y) \to 0$ as $y \to 0$, i.e. it tends to zero as $\lambda(\Omega)$ tends to zero (at least for the case of the Gaussian measure). In their result the dependence on $\lambda(\Omega)$ is precise, whereas the dependence on $\gamma^n(\Omega)$ is not explicit. We focus on the one dimensional case: in this setting Theorem 1.2 of [4] gives that

$$P_\gamma(\Omega) \geq J_\gamma(\gamma(\Omega)) + \frac{\lambda(\Omega)}{C(\gamma(\Omega))}\sqrt{\log\left(1/\lambda(\Omega)\right)}, \tag{0.1}$$

where $C(\gamma(\Omega))$ is a constant that depends only on $\gamma(\Omega)$. In this paper, Theorem 2.10 is a version of this statement which is actually valid for all symmetric log-concave measures μ on the real line. In addition, when the measure μ is not exponential-like (see Section 3), this quantitative inequality implies that a set of given measure and almost minimal perimeter is necessarily "close" to be a half-line, i.e. an isoperimetric set.

Organization of the paper

The outline of the paper is as follows: the first section recalls basic properties of the log-concave measures and the definition of the asymmetry. The second part gives the main tool, named the shifting lemma, and establishes a sharp quantitative isoperimetric inequality in Theorem 2.10. Moreover, we provide (slightly weaker) estimates invoking only classical functions. The last section is devoted to prove that non-exponential measures are, in a suitable sense, stable. As a matter of fact, we prove a continuity lemma: the asymmetry goes to zero as the isoperimetric deficit goes to zero.

1. The isoperimetric inequality on the real line

In this section, we recall the standard isoperimetric inequality for the log-concave measures, and the definition of the asymmetry which measures the gap between a given set and the sets of minimal perimeter. Let $\mu = f \cdot \mathcal{L}^1$ be a measure with density function f with respect to the 1-dimensional Lebesgue measure. Throughout this paper, we assume that

(i) f is supported and positive over some interval (a_f, b_f), where a_f and b_f can be infinite,

(ii) μ is a probability measure: $\int_{\mathbb{R}} f = 1$,

(iii) μ is a log-concave measure:

$$\forall\, x, y \in (a_f, b_f), \quad \forall\, \theta \in (0,1), \quad f(\theta x + (1-\theta)y) \geq f(x)^\theta f(y)^{1-\theta},$$

(iv) and μ is symmetric with respect to the origin:

$$\forall\, x \in \mathbb{R}, \quad f(x) = f(-x).$$

Remark 1.1. Observe that (iv) is not restrictive. As a matter of fact, the measure $\mu(. + \alpha)$, where $\alpha \in \mathbb{R}$, shares the same isoperimetric properties as the measure μ. By the same token, Assumption (ii) is obviously not restrictive.

We recall the definition of the μ-perimeter. Denote by Ω a measurable set. Define the set Ω^d of all points with density exactly $d \in [0,1]$ as

$$\Omega^d = \left\{ x \in \mathbb{R}, \quad \lim_{\rho \to 0} \frac{\mathcal{L}^1(\Omega \cap B_\rho(x))}{\mathcal{L}^1(B_\rho(x))} = d \right\},$$

where $B_\rho(x)$ denotes the ball with center x and radius ρ. Define the essential boundary $\partial^M \Omega$ as the set $\mathbb{R} \setminus (\Omega^0 \cup \Omega^1)$. Define the μ-perimeter as

$$P_\mu(\Omega) = \mathcal{H}^0_\mu(\partial^M \Omega) = \int_{\partial^M \Omega} f(x)\, \mathrm{d}\mathcal{H}^0(x), \tag{1.1}$$

where $\mathcal{H}^0$ is the *Hausdorff measure* of dimension 0 over $\mathbb{R}$ and $\mathcal{H}^0_\mu := f \cdot \mathcal{H}^0$. The isoperimetric function I_μ of the measure μ is defined by

$$I_\mu(r) = \inf_{\mu(\Omega) = r} P_\mu(\Omega). \tag{1.2}$$

Under Assumption (*iii*), we can give an explicit form to the isoperimetric function using a so-called function J_μ. Indeed, denote F the distribution function of the measure μ. Since the function f is supported and positive over some interval (a_f, b_f) then the cumulative distribution function is increasing on the interval (a_f, b_f). Define

$$J_\mu(r) = f(F^{-1}(r)), \tag{1.3}$$

where r is such that $0 < r < 1$, $J_\mu(0) = J_\mu(1) = 0$, and F^{-1} denotes the inverse function of F. Following the article [2] of S. G. Bobkov, since the measure μ is symmetric with respect to the origin, then the inverse function of F satisfies,

$$F^{-1}(r) = \int_{1/2}^{r} \frac{\mathrm{d}t}{J_\mu(t)}, \quad \forall r \in (0,1). \tag{1.4}$$

Using (1.4), one can check [2] that the following lemma holds.

Lemma 1.2. *The measure μ is log-concave* ***if and only if*** *J_μ is concave on* $(0,1)$.

Furthermore, it is known [3] that the infima of (1.2) are exactly (up to a μ-negligible set) the intervals $(-\infty, \sigma_-)$ and $(\sigma_+, +\infty)$, where $\sigma_- = F^{-1}(r)$ and $\sigma_+ = F^{-1}(1-r)$. The isoperimetric inequality states

$$P_\mu(\Omega) \geq J_\mu(\mu(\Omega)), \tag{1.5}$$

where Ω is a Lebesgue measurable set. This shows that, in the log-concave case, the isoperimetric function coincides with the function J_μ.

The asymmetry

We concern with quantifying the difference between any measurable set Ω and an isoperimetric infimum (i.e. measurable set such that the isoperimetric inequality (1.5) is an equality). Following [4], define the **asymmetry** $\lambda(\Omega)$ of a set Ω as

$$\lambda(\Omega) = \min\left\{\mu(\Omega\Delta(-\infty,\sigma_-))\,,\ \mu(\Omega\Delta(\sigma_+,+\infty))\right\}, \tag{1.6}$$

where $\sigma_- = F^{-1}(\mu(\Omega))$ and $\sigma_+ = F^{-1}(1-\mu(\Omega))$, and Δ is the symmetric difference operator.

Remark 1.3. The name asymmetry [5] is inherited from the case of the Lebesgue measure on $\mathbb{R}^n$. In this case, the sets with minimal perimeter are balls, hence very symmetric.

Define the **isoperimetric projection** of a set Ω as the open half-line achieving the minimum in (1.6). In the case where this minimum is not unique we can choose whatever infima as an isoperimetric projection.

2. Sharp quantitative isoperimetric inequalities

This section gives a sharp improvement of (1.5) involving the asymmetry $\lambda(\Omega)$. In [4], the authors use a technical lemma (Lemma 4.7, Continuity Lemma) to complete their proof. Their lemma applies in the n-dimensional case and is based on a compactness argument derived from powerful results in geometric measure theory. In the one-dimensional case, our approach is purely geometric and does not involve the continuity lemma.

2.1. The shifting lemma

The shifting lemma plays a key role in our proof. This lemma was introduced in [4] for the Gaussian measure. It naturally extends to even log-concave probability measures. For sake of readability, we begin with the shifting property.

Definition 2.1 (The shifting property)**.** We say that a measure ν satisfies the **shifting property** when for every open interval (a,b), the following is true:

- If $a+b \geq 0$ then for every (a',b') such that $a \leq a' < b' \leq +\infty$ and $\nu((a,b)) = \nu((a',b'))$, it holds $P_\nu((a,b)) \geq P_\nu((a',b'))$. In other words, if an interval is more to the right of 0, shifting it to the right with fixed measure, does not increase the perimeter.
- If $a+b \leq 0$ then for every (a',b') such that $-\infty \leq a' < b' \leq b$ and $\nu((a,b)) = \nu((a',b'))$, it holds $P_\nu((a,b)) \geq P_\nu((a',b'))$. In other words, if an interval is more to the left of 0, shifting it to the left with fixed measure, does not increase the perimeter.

Remark 2.2. As the perimeter is complement-invariant, we may also shift "holes". The shifting property is equivalent to the following property.

- If $a+b \geq 0$ then for every (a',b') such that $a \leq a' < b' \leq +\infty$ and $\nu((a,b)) = \nu((a',b'))$, it holds $P_\nu((-\infty,a) \cup (b,+\infty)) \geq P_\nu((-\infty,a') \cup (b',+\infty))$.

- If $a+b \leq 0$ then for every (a', b') such that $-\infty \leq a' < b' \leq b$ and $\nu((a,b)) = \nu((a',b'))$, it holds $P_\nu((-\infty,a)\cup(b,+\infty)) \geq P_\nu((-\infty,a')\cup(b',+\infty))$.

Roughly, the next lemma shows that, for all measures such that Assumptions (i), (ii), and (iv) hold, Assumption (iii) is equivalent to the shifting property.

Lemma 2.3 (The shifting lemma)**.** *Every log-concave probability measure symmetric with respect to the origin has the shifting property.*

Conversely, let f be a continuous function, positive on an open interval and null outside. If the probability measure with density function f is symmetric with respect to the origin and enjoys the shifting property then it is log-concave.

Proof. Let x, r be in $(0,1)$ and t be in $(r/2,\ 1-r/2)$. Define $\varphi(t) = J_\mu(t-r/2) + J_\mu(t+r/2)$. It represents the μ-perimeter of $(F^{-1}(t-r/2), F^{-1}(t+r/2))$ with measure equal to r. The function J_μ is symmetric with respect to $1/2$ since the density function f is supposed to be symmetric. As the function J_μ is concave and symmetric with respect to $1/2$, so is the function φ. In particular φ is non-decreasing on $(r/2, 1/2]$ and non-increasing on $[1/2, 1-r/2)$. This gives the shifting property.
Conversely, let f be a continuous function, positive on an open interval and null outside. Define the isoperimetric function J_μ as in (1.3). We recall that μ is log-concave if and only if J_μ is concave on $(0,1)$. Since the function J_μ is continuous, it is sufficient to have $J_\mu(x) \geq (1/2)\left(J_\mu(x-d) + J_\mu(x+d)\right)$, for all $x \in (0,1)$, where d is small enough to get $x-d \in (0,1)$ and $x+d \in (0,1)$. Let x and d be as in the previous equality. Since μ is symmetric, assume that $x \leq 1/2$. Put $a = F^{-1}(x)$, $b = F^{-1}(1-x)$, $a' = F^{-1}(x+d)$, $b' = F^{-1}(1-x+d)$, then (a',b') is a shift to the right of (a,b). By the shifting property, we get $P_\mu((a,b)) \geq P_\mu((a',b'))$. The function J_μ is symmetric with respect to $1/2$, it yields (see Figure 2.1),

$$\begin{aligned} P_\mu((a,b)) &= J_\mu(x) + J_\mu(1-x) &&= 2J_\mu(x), \\ P_\mu((a',b')) &= J_\mu(x+d) + J_\mu(1-x+d) &&= J_\mu(x+d) + J_\mu(x-d). \end{aligned}$$

This ends the proof. □

The key idea of the previous lemma is based on standard properties of the concave functions. Nevertheless, it is the main tool to derive quantitative isoperimetric inequalities. We see that the "shifting property" is particular to the one dimensional case and do not extend to higher dimensions.

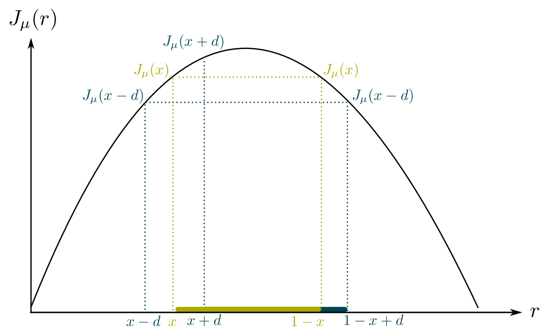

Figure 2.1. The log-concavity is equivalent to the shifting property

2.2. **Lower bounds on the perimeter**

We now recall a result on the structure of sets with finite perimeter on the real line.

Lemma 2.4. *Let Ω be a set of finite μ-perimeter. Then*

$$\Omega = \Big(\bigcup_{n\in I}(a_n, b_n)\Big)\bigcup\mathcal{E},$$

where I is at most countable, $\mathcal{E}$ such that $\mu(\mathcal{E}) = 0$, and (a_n, b_n) such that

$$d\Big((a_n, b_n), \bigcup_{k\in I\setminus\{n\}}(a_k, b_k)\Big) > 0, \tag{2.1}$$

for all n in I.

Proof. Consider $(K_k)_{k\in\mathbb{N}}$ a sequence of compact sets such that, for all $k \geq 0$, $K_0 \subset \ldots \subset K_k \subset (-a_f, a_f)$ and $\cup_{k\in\mathbb{N}}K_k = (-a_f, a_f)$. Then, it yields

$$\Omega = \Big(\bigcup_{k\in\mathbb{N}}(\Omega\cap K_k)\Big)\bigcup E, \tag{2.2}$$

where E is such that $\mu(E) = 0$. Let k be an integer. On the compact K_k the function f is bounded from below by a positive real. Thus if $\Omega\cap K_k$ has finite μ-perimeter, it also has finite perimeter. As mentioned in [1], one knows that every set with finite Lebesgue perimeter can be written as at most countable union of open intervals and a set of measure equal to zero. It holds

$$\Omega \cap K_k = \Big(\bigcup_{n\in I_k} (a_n, b_n)\Big) \bigcup \mathcal{E}_k,$$

where I_k is at most countable, $\mathcal{E}_k$ is such that $\mu(\mathcal{E}_k) = 0$, and (a_n, b_n) is such that

$$d\Big((a_n, b_n), \bigcup_{l\in I_k\setminus\{n\}} (a_l, b_l)\Big) > 0, \tag{2.3}$$

for all n in I_k and d the euclidean distance over the real line. Denote $\mathbb{1}_\Omega$ the indicator function of Ω and $\mathbb{1}'_\Omega$ its distributional derivative. The property (2.3) is a consequence of the fact that $\mathbb{1}'_\Omega$ is locally finite). Since K_k is compact, the set I_k is finite. One can check that the decomposition (2.2) gives the result. □

The newt lemma shows that among sets of given measure and given asymmetry, the intervals or complements of intervals have minimal perimeter.

Lemma 2.5. *Let Ω be a measurable set with μ-measure at most $1/2$ and $\lambda(\Omega)$ be the asymmetry of Ω. Then, it holds*

$$P_\mu(\Omega) \geq \min\{P_\mu\,(\Omega_c)\,,\, P_\mu\,(\Omega_d)\,\},$$

where

- $\Omega_c = \big(F^{-1}\big(\frac{\lambda(\Omega)}{2}\big)\,,\; F^{-1}\big(\mu(\Omega) + \frac{\lambda(\Omega)}{2}\big)\big)$,
- $\Omega_d = \big(-\infty\,,\; F^{-1}\big(\mu(\Omega) - \frac{\lambda(\Omega)}{2}\big)\big) \bigcup \big(F^{-1}\big(1 - \frac{\lambda(\Omega)}{2}\big)\,,\; +\infty\big)$,

are sets such that $\lambda(\Omega_c) = \lambda(\Omega_d) = \lambda(\Omega)$ and $\mu(\Omega_c) = \mu(\Omega_d) = \mu(\Omega)$.

Let us emphasize that the sets Ω_c and Ω_d have fixed isoperimetric projection (i.e. $(-\infty, -\sigma)$), asymmetry, and measure. Observe that these properties are satisfied only for particular values of $\mu(\Omega)$ and $\lambda(\Omega)$.

Proof. For sake of readability, the proof can be found in Appendix A. □

In the following, we describe the conditions on $(\mu(\Omega,\, \lambda(\Omega))$ for which the sets Ω_c and Ω_d exist. The next lemma shows that asymmetry and perimeter are complement invariant.

Lemma 2.6. *The symmetric difference, the asymmetry, and the perimeter are complement-invariants. Moreover, it holds $m(A) = m(A^c)$ where*

$$m(A) = \min\left\{\mu(A),\; 1 - \mu(A)\right\}.$$

Proof. Remark that $\mathbb{1}_{A\Delta B} = |\mathbb{1}_A - \mathbb{1}_B|$, it follows that the symmetric difference is complement-invariant. The essential boundary is complement-invariant, thus Definition 1.1 shows that the μ-perimeter is complement-invariant. Considering the symmetry of the isoperimetric function J_μ, we claim that the isoperimetric projections are complements of the isoperimetric projections of the complement. This latter property and the fact that the symmetric difference is complement-invariant give that the asymmetry is complement-invariant. The last equality is easy to check since μ is a probability measure. □

Consider the domain $D = \{(\mu(\Pi), \lambda(\Pi)),\ \Pi \text{ measurable set}\}$. Since the asymmetry is complement-invariant, the domain D is symmetric with respect to the axis $x = 1/2$. Furthermore, we have the next lemma.

Lemma 2.7. *It holds* $0 \leq \lambda(\Pi) \leq \min(2\,m(\Pi)\,,\,1 - m(\Pi))$*, where* Π *is a measurable set, and* $m(\Pi) = \min\{\mu(\Pi)\,,\ 1 - \mu(\Pi)\}$.

Proof. Let Π be a measurable set. As asymmetry $\lambda(\Pi)$ and $m(\Pi)$ are complement-invariant (see Lemma 2.6), suppose that $\mu(\Pi) \leq 1/2$ thus $m(\Pi) = \mu(\Pi)$. Using symmetry with respect to the origin, suppose that $(-\infty, -\sigma)$ is an isoperimetric projection of Π (where $\sigma = -F^{-1}(\mu(\Pi))$.
We begin with the inequality $\lambda(\Pi) \leq 1 - \mu(\Pi)$. Since $(-\infty, -\sigma)$ is an isoperimetric projection of Π, it holds

$$\mu(\Pi \cap (\sigma, +\infty)) \leq \mu(\Pi \cap (-\infty, -\sigma)) = \mu(\Pi) - \lambda(\Pi)/2.$$

Remark that $\mu((-\sigma, \sigma)) = 1 - 2\,\mu(\Pi)$. Hence, $\lambda(\Pi)/2 = \mu(\Pi \cap (-\sigma, +\infty)) \leq 1 - 2\,\mu(\Pi) + \mu(\Pi) - \lambda(\Pi)/2$, which gives the expected result.
The inequality $\lambda(\Pi) \leq 2\,\mu(\Pi)$ can be deduced from

$$\lambda(\Pi)/2 = \mu((-\infty, -\sigma) \setminus \Pi) \text{ and } \mu((-\infty, -\sigma) \setminus \Pi) \leq \mu((-\infty, -\sigma)) = \mu(\Pi).$$

It is clear that $\lambda(\Pi) \geq 0$, this ends the proof. □

Lemma 2.8. *Let* Ω *be a measurable set with* μ*-measure at most* $1/2$ *and* $\lambda(\Omega)$ *be the asymmetry of* Ω*. Then*

- *the connected set of the form*

$$\Omega_c = \left(F^{-1}(\lambda(\Omega)/2)\,,\ F^{-1}(\mu(\Omega) + \lambda(\Omega)/2)\right)$$

  satisfies $\mu(\Omega_c) = \mu(\Omega)$ *and* $\lambda(\Omega_c) = \lambda(\Omega)$ *when* $0 < \lambda(\Omega) \leq 1 - \mu(\Omega)$*,*

- *and the disconnected set of the form*

$$\Omega_d = (-\infty\,,\ F^{-1}(\mu(\Omega) - \lambda(\Omega)/2)) \cup (F^{-1}(1 - \lambda(\Omega)/2)\,,\ +\infty)$$

satisfies $\mu(\Omega_d) = \mu(\Omega)$ *and* $\lambda(\Omega_d) = \lambda(\Omega)$ *when* $0 < \lambda(\Omega) \leq \mu(\Omega)$.

Besides, when $0 < \lambda(\Omega) \leq \mu(\Omega)$, $P_\mu(\Omega_d) \leq P_\mu(\Omega_c)$ *with equality* ***if and only if*** $\mu(\Omega) = 1/2$.

Proof. By construction (see Appendix A), the sets Ω_c and Ω_d verify three properties:

(1) their measure is $\mu(\Omega)$,

(2) their asymmetry is $\lambda(\Omega)$,

(3) their isoperimetric projection is $(-\infty, -\sigma)$.

We recall that $\mu(\Omega) \leq 1/2$. Using Lemma 2.7, it is easy to check that Ω_c satisfies these properties if and only if

$$0 \leq \lambda(\Omega) \leq \min(2\,\mu(\Omega),\, 1 - \mu(\Omega)). \tag{2.4}$$

Using the definition of the isoperimetric projection, one can check that Ω_d satisfies these properties if and only if

$$0 \leq \lambda(\Omega) \leq \mu(\Omega). \tag{2.5}$$

Notice that on domain $0 \leq \lambda(\Omega) \leq \mu(\Omega)$ both sets exist. On this domain,

$$P_\mu(\Omega_d) - P_\mu(\Omega_c) = J_\mu(\mu(\Omega) - \lambda(\Omega)/2) - J_\mu(\mu(\Omega) + \lambda(\Omega)/2).$$

Since $\mu(\Omega) - \lambda(\Omega)/2 \leq \mu(\Omega) + \lambda(\Omega)/2 \leq 1 - \mu(\Omega) + \lambda(\Omega)/2$, we deduce from the concavity and the symmetry of the isoperimetric function that $P_\mu(\Omega_d) \leq P_\mu(\Omega_c)$ with equality if and only if $\mu(\Omega) = 1/2$. Using (2.4) and (2.5), we conclude the proof. □

We are concerned with an upper bound on the asymmetry of sets of given measure and given perimeter. Define the **isoperimetric deficit** of Ω as

$$\delta_\mu(\Omega) = P_\mu(\Omega) - J_\mu(\mu(\Omega)). \tag{2.6}$$

Define the **isoperimetric deficit function** K_μ as follows.

- On $0 < y \leq x \leq 1/2$, set $K_\mu(x,\, y) = J_\mu\,(x - y/2) - J_\mu\,(x) + J_\mu\,(y/2)$.

- On $0 < x \leq 1/2$ and $x < y \leq \min(2x,\, 1-x)$, set
$$K_\mu(x,\, y) = J_\mu\,(x + y/2) - J_\mu\,(x) + J_\mu\,(y/2)\,.$$

The isoperimetric deficit function $K_\mu(x,y)$ is defined on the domain of all the possible values of $(m(\Omega), \lambda(\Omega))$ (see Lemma 2.7). The next lemma focuses on the variations of K_μ.

Lemma 2.9. *Let $0 < x \leq 1/2$. The function $y \mapsto K_\mu(x,\, y)$ is a non-decreasing lower semi-continuous function. Besides, it is concave on $x < y \leq \min(2x,\, 1-x)$.*

Proof. The proof is essentially based on the concavity of J_μ.

On $0 < y \leq x$**:** Let $\Psi(t) = 1/2\,(J_\mu\,(x/2 - t) + J_\mu\,(x/2 + t))$. Then the point $(x/2,\, \Psi(t))$ is the middle of the chord joining $(x/2 - t,\, J_\mu(x/2 - t))$ and $(x/2 + t,\, J_\mu(x/2 + t))$. Since J_μ is concave, it is well known that Ψ is a non-increasing function. Remark that $K_\mu(x,y) = 2\,\Psi(x/2 - y/2) - J_\mu\,(x)$, thus $y \mapsto K_\mu(x,\, y)$ is non-decreasing. Moreover the function is continuous as sum of continuous functions.

On $x < y \leq \min(2x,\, 1-x)$**:** The function $y \mapsto K_\mu(x,\, y)$ is clearly concave as sum of two concave functions (thus continuous). On this domain,
$$(y/2) + (x + y/2) = x + y \leq x + \min(2x,\, 1-x) \leq 1.$$
Hence the interval $\omega_y = (F^{-1}(y/2), F^{-1}(x + y/2))$ is on the left of the origin. Remark that $K_\mu(x,y) = P_\mu(\omega_y) - J_\mu\,(x)$. The shifting lemma (Lemma 2.3) applies here and shows that the function $y \mapsto K_\mu(x,\, y)$ is non-decreasing (as y increases, ω_y shifts to the right).

The variation at x is given by $K_\mu\,(x, x^+) - K_\mu(x,x) = J_\mu\,(3/2\,x) - J_\mu\,(x/2)$, where $K_\mu\,(x, x^+) = \lim_{y \to x^+} K_\mu\,(x,y)$. One can check that $|1/2 - x/2| \geq |1/2 - 3x/2|$. Using the symmetry with respect to 1/2 and the concavity of J_μ, one can check that $J_\mu\,(3/2\,x) \geq J_\mu\,(x/2)$. Hence $K_\mu\,(x, x^+) \geq K_\mu(x,x)$.
This discussion shows that $y \mapsto K_\mu(x,\, y)$ is non-decreasing and lower semi-continuous on the whole domain. This ends the proof. ☐

Define the generalized inverse function of $y \mapsto K_\mu(x,\, y)$ as

$$K^{-1}_{\mu,\, x}(d) = \sup\left\{y \mid 0 \le y \le \min(2x,\, 1-x) \text{ and } K_\mu(x,y) \le d\right\}.$$

Lemma 2.9 shows that $y \mapsto K_\mu(x,\, y)$ is a non-decreasing lower semi-continuous function. It is easy to check that $K^{-1}_{\mu,\, x}$ is non-decreasing. The next theorem is the main result of this paper.

Theorem 2.10. *Let Ω be a measurable set and $\lambda(\Omega)$ be the asymmetry of Ω. Set $m(\Omega) = \min\left\{\mu(\Omega)\,,\ 1-\mu(\Omega)\right\}$, then*

$$\delta_\mu(\Omega) \ge K_\mu(m(\Omega),\, \lambda(\Omega))\,, \tag{2.7}$$

and this inequality is sharp. Moreover, it holds

$$\lambda(\Omega) \le K^{-1}_{\mu,\, m(\Omega)}(\delta(\Omega)). \tag{2.8}$$

Proof. Let Ω be a measurable set. If Ω has infinite μ-perimeter the result is true, hence assume that Ω has finite μ-perimeter. Then, it suffices to prove that

- If $0 < \lambda(\Omega) \le m(\Omega)$ then
$$P_\mu(\Omega) \ge J_\mu(m(\Omega) - \lambda(\Omega)/2) + J_\mu(\lambda(\Omega)/2), \tag{2.9}$$
- If $m(\Omega) < \lambda(\Omega) \le \min(2\, m(\Omega),\, 1-m(\Omega))$ then
$$P_\mu(\Omega) \ge J_\mu(m(\Omega) + \lambda(\Omega)/2) + J_\mu(\lambda(\Omega)/2), \tag{2.10}$$

and that these inequalities are sharp. We distinguish four cases as illustrated in Figure 2.2.

If Ω has measure at most $1/2$, then $m(\Omega) = \mu(\Omega)$. Consider sets Ω_c defined in (A.1) and Ω_d defined in (A), compute

$$\begin{aligned} P_\mu(\Omega_d) &= J_\mu(\mu(\Omega) - \lambda(\Omega)/2) + J_\mu(\lambda(\Omega)/2),\\ P_\mu(\Omega_c) &= J_\mu(\mu(\Omega) + \lambda(\Omega)/2) + J_\mu(\lambda(\Omega)/2). \end{aligned} \tag{2.11}$$

Lemma 2.5 says that Ω has μ-perimeter greater or equal than Ω_c or Ω_d.

Domain 1: If $\mu(\Omega) < \lambda(\Omega) \le 1 - \mu(\Omega)$ (and thus $m(\Omega) < \lambda(\Omega) \le 1 - m(\Omega)$) then from Lemma 2.8 we know that Ω_d does not exist for such range of asymmetry. Necessary, it follows that $P_\mu(\Omega) \ge P_\mu(\Omega_c)$. Using (2.11), we complete (2.10).

Domain 2: If $0 < \lambda(\Omega) \le \mu(\Omega)$ (and thus $0 < \lambda(\Omega) \le m(\Omega)$) then from Lemma 2.8 we know that $P_\mu(\Omega_d) \le P_\mu(\Omega_c)$. Thus $P_\mu(\Omega) \ge P_\mu(\Omega_d)$. Using (2.11), we get (2.9).

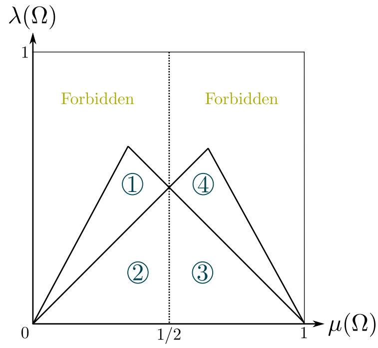

Figure 2.2. Domains of the sets with minimal perimeter given measure and asymmetry

If Ω has measure greater than $1/2$, then $1-\mu(\Omega) = m(\Omega)$. The Lemma 2.6 shows how to deal with sets of large measure and allows us to consider either Ω or its complement.

Domain 3: If $0 < \lambda(\Omega) \leq 1-\mu(\Omega)$ (and thus $0 < \lambda(\Omega) \leq m(\Omega)$), the complement of Ω satisfies $0 < \lambda(\Omega^c) \leq \mu(\Omega^c)$ (Domain 2). Thus we know that $P_\mu(\Omega_d) \leq P_\mu(\Omega_c)$ (see the previous case on Domain 2). Finally, $P_\mu(\Omega) \geq P_\mu\left(\Omega_d^c\right)$ where Ω_d has same asymmetry and measure equal to $m(\Omega)$. Using (2.11), we complete (2.9).

Domain 4: If $1 - \mu(\Omega) < \lambda(\Omega) \leq \mu(\Omega)$ (and thus $m(\Omega) < \lambda(\Omega) \leq 1-m(\Omega)$), the complement of Ω satisfies $\mu(\Omega^c) < \lambda(\Omega^c) \leq 1-\mu(\Omega^c)$ (Domain 1). From the case on Domain 1, we know that $P_\mu\left(\Omega^c\right) \geq P_\mu(\Omega_c)$. Thus, $P_\mu(\Omega) \geq P_\mu\left(\Omega_c^c\right)$ where Ω_c has same asymmetry and measure equal to $m(\Omega)$. Using (2.11), we get (2.10).

This case analysis shows (2.7). Set $x = m(\Omega)$, the upper bound (2.8) is a consequence of the definition of $K^{-1}_{\mu,\, x}$ and (2.7). This concludes the proof. $\square$

Remark 2.11. We focus on the Gaussian measure γ. Observe that

$$J_\gamma(t) \underset{t\to 0}{\sim} t\sqrt{2\log\left(1/t\right)},$$

so that

$$K_\gamma(x,y) \underset{y\to 0}{\sim} J_\gamma\left(\frac{y}{2}\right) \underset{y\to 0}{\sim} \frac{y}{2}\sqrt{2\log\left(2/y\right)}.$$

In particular, there exists a constant $C(x)$ that depends only on x such that

$$K_\gamma(x,y) \geq \frac{y}{C(x)}\sqrt{\log(1/y)}, \quad \text{with } 0 \leq y \leq \min(2x,\, 1-x).$$

Eventually, we recover (0.1) from Theorem 2.10.

The equalities (2.11) and the case analysis of the proof of Theorem 2.10 give the explicit lower bounds on μ-perimeter.

Proposition 2.12. *Given two positive numbers μ, λ, we consider the following penalized isoperimetric problem:*

$$\min\big\{P_\mu(\Omega)\ :\ \Omega \subseteq \mathbb{R},\ \text{with } \mu(\Omega)=\mu \text{ and } \lambda(\Omega)=\lambda\big\}. \tag{2.12}$$

Then the solution is given by the following sets (see Figure 2.2)

- $\Omega_c = \big(F^{-1}(\frac{\lambda}{2})\,,\ F^{-1}(\mu+\frac{\lambda}{2})\big)$, *with* $0<\mu<\lambda\leq 1-\mu$ *and* $\mu\leq 1/2$ *(Domain 1),*
- $\Omega_d = \big(-\infty\,,\ F^{-1}(\mu-\frac{\lambda}{2})\big)\cup\big(F^{-1}(1-\frac{\lambda}{2})\,,\ +\infty\big)$, *with* $0<\lambda\leq\mu$ *and* $\mu\leq 1/2$ *(Domain 2),*
- $\Omega_d^c = \big(F^{-1}(1-\mu-\frac{\lambda}{2})\,,\ F^{-1}(1-\frac{\lambda}{2})\big)$, *with* $0<\lambda\leq 1-\mu$ *and* $1/2\leq\mu<1$ *(Domain 3),*
- $\Omega_c^c = \big(-\infty\,,\ F^{-1}(\frac{\lambda}{2})\big)\bigcup\big(F^{-1}(1-\mu+\frac{\lambda}{2})\,,\ +\infty\big)$, *with* $1-\mu<\lambda\leq\mu$ *and* $1/2\leq\mu<1$ *(Domain 4).*

3. Stability of isoperimetric sets

In general the quantitative estimate of Theorem 2.10 is not a stability result, since one can have $\delta_\mu = 0$ and $\lambda > 0$ for suitable choices of μ. Indeed, consider the exponential case where $\mu = f\cdot\mathcal{L}^1$ with

$$f(t) = \frac{1}{2}\exp(-|t|), \quad \forall t\in\mathbb{R}.$$

It holds that

$$J_{\exp}(t) = t\,\mathbb{1}_{[0,1/2]} + (1-t)\,\mathbb{1}_{[1/2,1]}.$$

It yields that $K_{\exp} = 0$ on $0\leq y\leq x\leq 1/2$. Hence, there exists sets with a positive asymmetry and an isoperimetric deficit null. In the case of *the exponential-like distributions* (defined later on), the intervals

$(-\infty, F^{-1}(r))$ and $(F^{-1}(1-r), +\infty)$ are not the only sets with minimal perimeter (up to a set of measure equals to 0) given measure r.

We specify this thought defining a natural hypothesis ($\mathbb{H}$). Furthermore, we prove that the asymmetry goes to zero as the isoperimetric deficit goes to zero under ($\mathbb{H}$).

3.1. The hypothesis $\mathbb{H}$

We can get a better estimate on the asymmetry making another hypothesis. From now, suppose that the measure μ is such that

$$\exists\, \varepsilon > 0 \quad \text{s.t.} \quad t \mapsto J_\mu(t)/t \text{ is decreasing on } (0, \varepsilon). \tag{$\mathbb{H}$}$$

This hypothesis means that J_μ is non-linear in a neighborhood of the origin. We can be more specific introducing the property:

$$\exists\, \varepsilon > 0 \text{ and } c > 0 \quad \text{s.t.} \quad J_\mu(t) = c\,t,\ \forall\, t \in [0, \varepsilon]. \tag{$\overline{\mathbb{H}}$}$$

Since $t \mapsto J_\mu(t)/t$ is non-increasing, it is not difficult to check that ($\overline{\mathbb{H}}$) is the alternative hypothesis of ($\mathbb{H}$). Furthermore, **exponential-like** measure can be defined by the following property:

$$\exists\, \tau > 0 \text{ and } c, c' > 0 \text{ s.t. } f(t) = c' \exp(ct),\ \forall\, t \in (-\infty, \tau). \tag{$\mathcal{E}xp$}$$

Proposition 3.1. *The property ($\overline{\mathbb{H}}$) is equivalent to the property ($\mathcal{E}xp$).*

Proof. The proof is derived from the equality $(F^{-1})'(t) = 1/J_\mu(t)$, for all $t \in (0, 1)$ (see [2]). Suppose that the measure satisfies ($\overline{\mathbb{H}}$). Using the above equality for sufficiently small values of r, one can check that $F^{-1}(r) = \frac{1}{c}\log(r) + c''$, where c'' is a constant. Hence $F(x) = \exp(c(x - c'')) = \frac{c'}{c}\exp(cx)$, which gives the property ($\mathcal{E}xp$). Conversely, suppose that the measure satisfies ($\mathcal{E}xp$). A simple computation gives the property ($\overline{\mathbb{H}}$). □

Suppose that μ satisfies ($\overline{\mathbb{H}}$). It is not difficult to check that the sets (and their symmetric) $(-\infty, F^{-1}(r - s)) \cup (F^{-1}(1 - s), +\infty)$, for all $s \in (0, r)$, have minimal perimeter among all sets of given measure r such that $r \leq \varepsilon$. It would be natural to define the asymmetry with these sets.

3.2. The continuity theorem

In the following, we give a more convenient bound on the asymmetry. Define the function L_μ as follows.

- On $0 < y \leq x \leq 1/2$, set
$$L_\mu(x,\, y) = J_\mu\,(y/2) - y/(2x)\, J_\mu\,(x)\,.$$
- On $0 < x \leq 1/2$ and $x < y \leq \min(2x,\, 1-x)$, set
$$L_\mu(x,\, y) = J_\mu\,(y/2) - y/(2(1-x))\, J_\mu\,(x)\,.$$

We have the following lemma:

Lemma 3.2. *Let Ω be a measurable set and $\lambda(\Omega)$ be the asymmetry of Ω. Let $m(\Omega) = \min\{\mu(\Omega)\,,\ 1-\mu(\Omega)\}$ and $\delta_\mu(\Omega) = P_\mu(\Omega) - J_\mu(\mu(\Omega))$. It holds,*
$$\delta_\mu(\Omega) \geq L_\mu(m(\Omega),\, \lambda(\Omega)) \geq 0\,. \tag{3.1}$$

Proof. Since the asymmetry, the perimeter, the isoperimetric deficit, and $m(\Omega)$ are complement invariant, suppose that $m(\Omega) = \mu(\Omega) \leq 1/2$. Set $x = m(\Omega)$ and $y = \lambda(\Omega)$.

On $0 < y \leq x$**:** Set $t = y/(2x-y)$ then $x - y/2 = t\,y/2 + (1-t)\,x$. Since J_μ is concave, it holds
$$\begin{aligned}
K_\mu(x,y) &= J_\mu\left(x - \frac{y}{2}\right) - J_\mu\,(x) + J_\mu\left(\frac{y}{2}\right),\\
&\geq (1+t)J_\mu\left(\frac{y}{2}\right) - tJ_\mu\,(x)\,,\\
&= \frac{1}{1-y/2x}\left(J_\mu\left(\frac{y}{2}\right) - \frac{y}{2x}\,J_\mu\,(x)\right),\\
&\geq J_\mu\left(\frac{y}{2}\right) - \frac{y}{2x}\,J_\mu\,(x)\,.
\end{aligned}$$
As J_μ is concave, the function $t \mapsto J_\mu(t)/t$ is non-increasing and thus $(2/y)J_\mu(y/2) - (1/x)J_\mu\,(x) \geq 0$.

On $x < y \leq \min(2x,\, 1-x)$**:** Using symmetry with respect to $1/2$, remark that
$$\begin{aligned}
K_\mu(x,y) &= J_\mu\left(x + \frac{y}{2}\right) - J_\mu\,(x) + J_\mu\left(\frac{y}{2}\right)\\
&= J_\mu\left((1-x) - \frac{y}{2}\right) - J_\mu\,(1-x) + J_\mu\left(\frac{y}{2}\right)
\end{aligned}$$
Substituting x with $1-x$, the same calculus as above can be done.

This ends the proof. ☐

The lower bound given in Lemma 3.2 is the key tool of the proof of the continuity theorem. The hypothesis ($\mathbb{H}$) ensures that the distribution is non-exponential. It is the right framework dealing with continuity as shown in the next theorem.

Theorem 3.3 (Continuity for non-exponential distributions)**.** *Assume that the measure μ satisfies the assumption $\mathbb{H}$, then the asymmetry goes to zero as the isoperimetric deficit goes to zero.*

Proof. The proof is based on Lemma 3.2 and Theorem 2.10. Let $u, v \in (0,1)$, define $\rho(u,v) = J_\mu(u)/u - J_\mu(v)/v$. Suppose $u < v$. Since J_μ is concave, it is easy to check that if $\rho(u,v) = 0$, then $\forall\, u' \leq u$, $\rho(u',v) = 0$. In particular $\mathbb{H}$ implies that $\forall\, u < v\,,\ \rho(u,v) > 0$, for sufficiently small values of v. Remark that $L_\mu(x,y) = (y/2)\rho(y/2,x)$ if $0 < y \leq x$, and $L_\mu(x,y) = (y/2)\rho(y/2, 1-x)$ if $x < y \leq \min(2x,\, 1-x)$. Hence $\mathbb{H}$ implies that $L_\mu > 0$. Using Lemma 3.2, it yields that $K_\mu > 0$.

Finally, it is easy to check that if $K_\mu > 0$ then there exists a neighborhood of 0 such that K_μ is increasing. Taking a sufficiently small neighborhood if necessary, one can suppose that K_μ is continuous (the only point of discontinuity of K_μ is $y = x$). On this neighborhood, $K_{\mu,x}^{-1}$ is a continuous increasing function. Using (2.8), this gives the expected result. ☐

Roughly, a set of given measure and almost minimal boundary measure is necessarily close to be a half-line. Moreover we recover the following well-known result.

Corollary 3.4. *Assume that the measure μ satisfies the assumption $\mathbb{H}$, then the half-lines are the* ***unique*** *sets of given measure and minimal perimeter* (*up to a set of μ-measure null*)*.*

This last results ensure that the asymmetry (1.6) is the relevant notion speaking of the isoperimetric deficit under ($\mathbb{H}$).

As already said, the main argument of our result (Lemma 2.3) is peculiar of dimension 1. Nevertheless, it would be interesting to know whether one can extend the results of [4] to non-exponential log-concave measures also in higher dimensions or not.

Acknowledgments: The author thanks F. Barthe for invaluable discussions and unfailing support. The author thanks F. Maggi for communicating useful references to him. Last but not least, the author thanks an anonimous referee for its careful reading and its useful remarks.

Appendix A. Proof of Lemma 2.5

As mentioned in Lemma 2.4, assume that $\Omega = \bigcup_{n\in I}(a_n, b_n)$ where I is an at most countable set and (2.1) holds. Suppose that

- an isoperimetric projection of Ω is $(-\infty, \sigma_-)$ (using a symmetry with respect to the origin if necessary),
- and that the measure of Ω is at most $1/2$ (and we will see at the end of this section how to extend our result to larger measures).

Then the real number $\sigma_- = F^{-1}(\mu(\Omega))$ is non-positive. Denote $\sigma = -\sigma_-$. Since $\mathbb{1}'_\Omega$ is locally finite, there exists a finite number of sets (a_n, b_n) included in $(-\sigma, \sigma)$, it follows that

$$\Omega = \Big(\bigcup_{h\in\Lambda_-} A_h\Big) \cup I \cup \Big(\bigcup_{h=1}^{N_-} A'_h\Big) \cup \Big(\bigcup_{h=1}^{N_+} B'_h\Big) \cup J \cup \Big(\bigcup_{h\in\Lambda_+} B_h\Big),$$

where

- Λ_- and Λ_+ are at most countable sets;
- $A_h = (\alpha_{A_h}, \beta_{A_h})$ with $\beta_{A_h} \leq -\sigma$ (α_{A_h} can be infinite);
- I is either empty or of the form $I = (\alpha_I, \beta_I)$ with $\alpha_I \leq -\sigma < \beta_I$;
- A'_h is either empty or of the form $A'_h = (\alpha_{A'_h}, \beta_{A'_h})$ with $-\sigma < \alpha_{A'_h}$ and $\alpha_{A'_h} + \beta_{A'_h} < 0$;
- B'_h is either empty or of the form $B'_h = (\alpha_{B'_h}, \beta_{B'_h})$ with $\beta_{B'_h} < \sigma$ and $\alpha_{B'_h} + \beta_{B'_h} \geq 0$;
- J is either empty or of the form $J = (\alpha_J, \beta_J)$ with $\alpha_J < \sigma \leq \beta_J$;
- and B_h is either empty or of the form $B_h = (\alpha_{B_h}, \beta_{B_h})$ with $\alpha_{B_h} \geq \sigma$ (β_{B_h} can be infinite).

From Ω we build Ω_0 with same measure, same asymmetry, same isoperimetric projection, and lower or equal perimeter. Denote $L = \bigcup_{h\in\Lambda_-} A_h$ and $A_0 = (-\infty, \beta_{A_0})$ where $\beta_{A_0} = F^{-1}(\mu(L))$. Since $\mu(L) \leq \mu(\Omega)$, then $\beta_{A_0} \leq -\sigma$. Using the isoperimetric inequality (1.5) with L, it follows that $P_\mu(A_0) \leq P_\mu(L)$. The same reason gives that there exist a real number $\alpha_{B_0} \geq \sigma$ and a set $B_0 = (\alpha_{B_0}, +\infty)$ with lower or equal perimeter than $\cup_{h\in\Lambda_+} B_h$ (if non-empty). Shift to the left the intervals A'_h until they reach I or $-\sigma$. Shift to the right the intervals B'_h until they reach J or σ. The above operation did not change the amount of mass on left of $-\sigma$ and on the right of σ. We build a set Ω_0 with same asymmetry and same isoperimetric projection as Ω and lower or equal perimeter,

$$\Omega_0 = A_0 \cup I_0 \cup J_0 \cup B_0,$$

where

- $A_0 = (-\infty, \beta_0)$ with $\beta_{A_0} \leq -\sigma$;
- I_0 is either empty or of the form $I_0 = (\alpha_{I_0}, \beta_{I_0})$ with $\alpha_{I_0} \leq -\sigma < \beta_{I_0}$;
- J_0 is either empty or of the form $J_0 = (\alpha_{J_0}, \beta_{J_0})$ with $\alpha_{J_0} < \sigma \leq \beta_{J_0}$;
- and B_0 is either empty or of the form $B_0 = (\alpha_{B_0}, +\infty)$ with $\alpha_{B_0} > \sigma$.

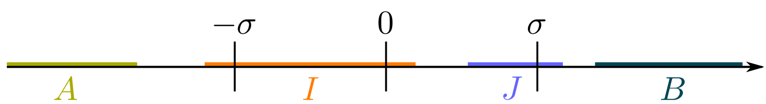

Figure A.1. The set Ω_0

A case analysis on the non-emptiness of sets I_0 and J_0 is required to obtain the claimed result. Every step described below lowers the perimeter (thanks to the shifting lemma, Lemma 2.3) and preserves the asymmetry. Before exposing this, we recall that the set Ω_0 is supposed to have $(-\infty, -\sigma)$ as an isoperimetric projection. Thus we pay attention to the fact that it is totally equivalent to ask either the asymmetry to be preserved or the quantity $\lambda(\Omega_0)/2 = \mu(\Omega_0 \cap (-\infty, -\sigma))$ to be preserved trough all steps described below.

If I_0 and J_0 are both nonempty: Applying a symmetry with respect to the origin if necessary, assume that the center of mass of the hole between I_0 and J_0 is not less than 0. We can shift this hole to the right until it touches σ. Using the isoperimetric inequality (1.5), assume that there exist only one interval of the form $(\alpha'_{B_0}, +\infty)$ on the right of σ. We get the case where I_0 is nonempty and J_0 is empty.

If I_0 is nonempty and J_0 is empty: Then shift the hole between A_0 and I_0 to the left until $-\infty$ (there exists one and only one hole between A_0 and I_0 since Ω_0 is not a full measure set of $(-\infty, -\sigma)$). We shift the hole between I_0 and B_0 to the right until $+\infty$ (one readily checks that its center of mass is greater than 0). We get the only interval with same asymmetry and same isoperimetric projection as the set Ω_0. This interval is of the form (the letter c stands for connected),

$$\Omega_c := \big(F^{-1}\left(\lambda(\Omega_0)/2\right)\,,\ F^{-1}\left(\mu(\Omega_0) + \lambda(\Omega_0)/2\right)\big). \tag{A.1}$$

If J_0 is nonempty and I_0 is empty: Shift to the right the hole between J_0 and B_0 to $+\infty$ (there exists one hole between J_0 and B_0 since Ω_0 is not a full measure set of $(\sigma, +\infty)$). We obtain a set $A_0 \cup J'$ where J' is a neighborhood of σ.

- If $\mu(J') > \mu(A_0)$, then shift J' to the right (which has center of mass greater than 0) till $J' \cap (\sigma, +\infty)$ has weight equal to $\mu(A_0)$ (in order to preserve asymmetry). Using a reflection in respect to the origin, we find ourselves in the case where I_0 is nonempty and J_0 is empty.
- If $\mu(J') \leq \mu(A_0)$, then shift J' (which has center of mass greater than 0) to the right until $+\infty$ and get the case where I_0 and J_0 are both empty.

If I_0 and J_0 are both empty: Then the set Ω_0 is of the form (d stands for disconnected),

$$\Omega_d = \big(-\infty, F^{-1}\left(\mu(\Omega_0) - \lambda(\Omega_0)/2\right)\big) \cup \big(F^{-1}\left(1 - \lambda(\Omega_0)/2\right), +\infty\big).$$

This concludes the proof.

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