# Non-asymptotic Tail Bounds for the Kostlan–Shub–Smale Field: Tensor PCA and Spherical $k$-Spin Complexity

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**Abstract**

This paper builds a hierarchy of explicit, non-asymptotic tail bounds for the supremum of the Kostlan–Shub–Smale (KSS) random field on the sphere, and applies it to two problems: Spiked Tensor PCA and the landscape of the spherical $k$-spin model. For Tensor PCA, we study the non-asymptotic statistical limits of estimating a rank-$R$ symmetric signal tensor of order $k \geq 3$ and dimension $d \geq 3$ from a single Gaussian observation at signal-to-noise ratio $\lambda$, through the *profile maximum likelihood estimator*, the MLE restricted to normalized rank-$R$ tensors of coherence at least $\kappa$. Our analysis uses a single reduction: a deterministic geometric inequality (the Tube Method) and a rank-reduction step bound the estimation error by the supremum of the canonical KSS field, which the Kac–Rice formula turns into a Gaussian integral against the expected absolute characteristic polynomial of a shifted Gaussian Orthogonal Ensemble, controlled in turn by the four explicit tail bounds of our hierarchy (three from a Mehta–Fyodorov representation, one from a Ben Arous–Dembo–Guionnet large deviation). The same reduction yields two results, each with explicit constants. For estimation, a finite-$(k, d)$ error bound recovers the asymptotically optimal rate $\sqrt{d \log k}$ of Perry, Wein and Bandeira, with explicit dependence on the rank $R$ and the coherence $\kappa$. For the landscape, a two-sided non-asymptotic bracketing of the annealed complexity of the spherical $k$-spin Hamiltonian recovers the Auffinger–Ben Arous–Černý complexity function in the high-dimensional limit.

# 1 Introduction

## 1.1 Tensor regression model and the profile MLE

Tensor regression generalizes linear regression to the setting where the response is a symmetric tensor $\boldsymbol{Y}$ of order $k \geq 2$ and dimension $d \geq 2$. The model reads

$$\boldsymbol{Y} \;=\; \lambda \boldsymbol{\sigma}^\star + \boldsymbol{W}\,, \tag{1.1}$$

where $\boldsymbol{\sigma}^\star$ is the signal tensor (with $\|\boldsymbol{\sigma}^\star\|_F = 1$), $\lambda > 0$ is the signal-to-noise ratio, and $\boldsymbol{W}$ is a standard Gaussian symmetric tensor with distribution $\mathbb{P}_0$ whose density is proportional to $\exp(-\|\boldsymbol{W}\|_F^2/2)$. Throughout, $\|\cdot\|_F$ denotes the Frobenius norm and $\langle \cdot, \cdot \rangle_{\mathcal{T}}$ the associated inner product on the space $\mathcal{T}(k, d)$ of symmetric tensors. We denote by $\mathfrak{S}_R$ the set of *normalized* symmetric tensors of rank at most $R$. Any $\boldsymbol{\sigma} \in \mathfrak{S}_R$ admits a rank-$R$ decomposition

$$\boldsymbol{\sigma} \;=\; \sum_{j=1}^{R} a_j\, t_j^{\otimes k}\,, \qquad a_j \neq 0\,, \quad \|\boldsymbol{\sigma}\|_F^2 = 1\,, \tag{1.2a}$$

where $t_j^{\otimes k}$ denotes the $k$-fold outer product of $t_j \in \mathbb{S}^{d-1}$ with itself, and the vectors $t_j$ are pairwise distinct.

The *coherence* of $\boldsymbol{\sigma}$ is defined by

$$\kappa^2(\boldsymbol{\sigma}) \;:=\; \max\Big\{\lambda_{\min}(\boldsymbol{G}) \,\Big|\, G_{ij} = \langle t_i, t_j\rangle^k \text{ and (1.2a) holds}\Big\}\,, \tag{1.2b}$$

where the maximum is taken over all rank-$R$ decompositions of $\boldsymbol{\sigma}$. The coherence satisfies $\kappa(\boldsymbol{\sigma}) \in (0,1]$, with $\kappa = 1$ when the components are pairwise orthogonal. Small values of $\kappa$ correspond to near-collinear components, which are statistically harder to disentangle.

**Profile Maximum Likelihood Estimator** The *profile Maximum Likelihood Estimator* (profile MLE) maximizes the inner product with the observation $\boldsymbol{Y}$ over the slice of $\mathfrak{S}_R$ on which the coherence is at least $\kappa$:

$$\hat{\lambda} \;=\; \langle \boldsymbol{Y}, \hat{\boldsymbol{\sigma}}\rangle_{\mathcal{T}}\,, \qquad \hat{\boldsymbol{\sigma}} \;\in\; \arg\max\Big\{\langle \boldsymbol{Y}, \boldsymbol{\sigma}\rangle_{\mathcal{T}} \,\Big|\, \boldsymbol{\sigma} \in \mathfrak{S}_R,\ \kappa(\boldsymbol{\sigma}) \geq \kappa\Big\}\,, \tag{1.3}$$

and $\hat{\boldsymbol{\sigma}}$ is well defined: the feasible set

$$\mathcal{C}_{R,\kappa} := \Big\{\boldsymbol{\sigma} \in \mathfrak{S}_R \,\Big|\, \kappa(\boldsymbol{\sigma}) \geq \kappa\Big\}$$

is *compact*: bounded in the unit sphere of $\mathcal{T}(k,d)$, and closed because the coherence floor $\kappa(\boldsymbol{\sigma}) \geq \kappa > 0$ bounds the coefficients ($\|\boldsymbol{a}\|_2 \leq 1/\kappa$, Lemma 9) while the components range over the compact sphere $\mathbb{S}^{d-1}$, ruling out the border-rank degenerations through which $\{\text{rank} \leq R\}$ fails to be closed for $k \geq 3$. The continuous Gaussian functional $\boldsymbol{\sigma} \mapsto \langle \boldsymbol{Y}, \boldsymbol{\sigma}\rangle_{\mathcal{T}}$ therefore attains an almost surely unique maximum on $\mathcal{C}_{R,\kappa}$ (Lifshits, 1983; Tsirelson, 1976, Theorem 3); concentrating out $\lambda$ in the Gaussian log-likelihood of (1.1) leaves exactly (1.3), whence the name *profile MLE*.

## 1.2 Geometric reduction and the Kac–Rice integral

A deterministic geometric inequality (Lemma 8), referred to as the *Tube Method*, controls the estimation error by the supremum of the empirical noise process on the feasible manifold:

$$\|\hat{\boldsymbol{\sigma}} - \boldsymbol{\sigma}^\star\|_F^2 \;\leq\; \frac{4\,\Gamma_{R,\kappa}}{\lambda}\,, \qquad \Gamma_{R,\kappa} := \sup\Big\{|\langle \boldsymbol{W}, \boldsymbol{\sigma}\rangle_{\mathcal{T}}| \,\Big|\, \boldsymbol{\sigma} \in \mathcal{C}_{R,\kappa}\Big\}\,,$$

with $\Gamma_{R,\kappa}$ referred to as the *noise level* and, by continuity of $\boldsymbol{\sigma} \mapsto \langle \boldsymbol{W}, \boldsymbol{\sigma}\rangle_{\mathcal{T}}$ and compactness of $\mathcal{C}_{R,\kappa}$, this supremum is attained. The rank-reduction inequality (see Lemma 9) then bounds

$$\Gamma_{R,\kappa} \leq \frac{\sqrt{R}}{\kappa}\,\Gamma_{1,1}\,, \qquad \Gamma_{1,1} \;=\; \sup_{\boldsymbol{\theta} \in \mathbb{S}^{d-1}} \Big\{|X(\boldsymbol{\theta})|\Big\}\,, \qquad X(\boldsymbol{\theta}) := \langle \boldsymbol{W}, \boldsymbol{\theta}^{\otimes k}\rangle_{\mathcal{T}}\,, \tag{1.4a}$$

where the covariance $\mathbb{E}[X(\boldsymbol{\theta})X(\boldsymbol{v})] = \langle \boldsymbol{\theta}, \boldsymbol{v}\rangle^k$ identifies $X$ as the canonical *Kostlan–Shub–Smale* (KSS) random field on the sphere. This is the field named in the title: every bound below concerns the upper tail of its supremum, for which we write

$$\widetilde{\Gamma}_{1,1}(u) \;:=\; \mathbb{P}\Big\{\sup_{\boldsymbol{\theta} \in \mathbb{S}^{d-1}} X(\boldsymbol{\theta}) > u\Big\}\,. \tag{1.4b}$$

The two-sided supremum $\Gamma_{1,1}$ of (1.4a) follows from the symmetry $X \overset{d}{=} -X$ of the centred field:

$$\mathbb{P}\{\Gamma_{1,1} > u\} \;\leq\; 2\widetilde{\Gamma}_{1,1}(u)\,; \tag{1.4c}$$

the factor 2 is the only cost of passing to the two-sided sup, and appears in the failure probability of Theorem 1.

**Kac–Rice integral.** Throughout, the *Gaussian Orthogonal Ensemble* of size $n$, denoted GOE($n$), is the law of a real symmetric $n \times n$ random matrix $G = (G_{ij})$ whose entries are jointly Gaussian, centred, independent up to symmetry, with the Mehta normalization $\text{var}(G_{ii}) = 1$ and $\text{var}(G_{ij}) = 1/2$ for $i \neq j$; equivalently, the density on the space of real symmetric matrices is proportional to $\exp(-\text{tr}(G^2)/2)$ and the law is invariant under conjugation by any orthogonal matrix (Mehta, 2004). The Kac–Rice formula (Azaïs and Wschebor, 2009), combined with the conditional GOE law of the Riemannian Hessian (Auffinger et al., 2013, Lemma 3.2(b)), reduces $\widetilde{\Gamma}_{1,1}(u)$ to a Gaussian integral against the expected absolute characteristic polynomial of a shifted GOE matrix. Defining

$$\delta_0(u) \;:=\; C_{k,d} \int_u^{\infty} \mathbb{E}\big[|\det(G_{d-1} - \rho x\, I_{d-1})|\big]\, \varphi(x)\, dx\,, \tag{1.4d}$$

the Kac–Rice formula gives

$$\widetilde{\Gamma}_{1,1}(u) \;\le\; \delta_0(u)\,, \tag{1.4e}$$

with constants

$$\rho := \sqrt{\frac{k}{2(k-1)}}\,, \qquad C_{k,d} := \frac{2\sqrt{\pi}}{\Gamma(d/2)}\,(k-1)^{\frac{d-1}{2}}\,, \qquad G_{d-1} \sim \mathrm{GOE}(d-1) \tag{1.4f}$$

and $\varphi$ the standard normal density. It is proved in (Azaïs and Wschebor, 2009, Theorem 8.12) that the bound (1.4e) is super-exponentially precise as $u \to \infty$, so that $\delta_0$ is asymptotically the right scale to describe $\widetilde{\Gamma}_{1,1}$.

Section 2 makes $\delta_0$ explicit: on $[u_{\mathrm{IMF}}, \infty)$ with $u_{\mathrm{IMF}} := \sqrt{2d-1}/\rho$ (Szegő's bound on the largest root of $H_{d-1}$, Lemma 3), it admits the exact closed form $\delta_{\mathrm{exact}}$ via the Mehta–Fyodorov representation (Theorem 2), with $\delta_0 = \delta_{\mathrm{exact}}$ pointwise there; the relaxations $\delta_{\mathrm{IMF}}, \delta_{\mathrm{SMF}}, \delta_{\mathrm{SM}}$ are explicit upper bounds on appropriate sub-ranges.

**Asymptotic baseline as $u \to \infty$, with $d$ fixed.** Replacing the expected determinant in (1.4d) by its leading monomial $(\rho x)^{d-1}$ gives, by Lemma 12, the asymptotic equivalent

$$\delta_{\mathrm{bl}}(u) \;:=\; \frac{\sqrt{2}}{\Gamma(d/2)} \left(\frac{k}{2}\right)^{\frac{d-1}{2}} u^{d-2}\, e^{-u^2/2}\,, \qquad \delta_0(u) \;\sim\; \delta_{\mathrm{bl}}(u) \quad \text{as } u \to \infty \text{ with } d \text{ fixed}\,. \tag{1.5}$$

This is the optimal tail rate against which the non-asymptotic bounds are measured; $\delta_{\mathrm{bl}}$ is not itself an upper bound on $\delta_0$, only the asymptotic equivalent.

### 1.3 Hierarchy of four explicit tail bounds

We develop a hierarchy of four explicit non-asymptotic upper bounds on $\delta_0(u)$ and, by (1.4d), on the excursion probability $\mathbb{P}\big\{\sup_{\boldsymbol{\theta}\in\mathbb{S}^{d-1}} X(\boldsymbol{\theta}) > u\big\}$, each valid above an explicit threshold:

$$u_{\mathrm{IMF}} := \frac{\sqrt{2d-1}}{\rho}\,, \qquad u_{\mathrm{SMF}} := 2\sqrt{d}\,, \qquad u_{\mathrm{SM}} := \frac{32\sqrt{d-1}}{\rho}\,.$$

For $k, d \ge 3$ one has $u_{\mathrm{IMF}} \le u_{\mathrm{SMF}} \le u_{\mathrm{SM}} \le 32\sqrt{2d}$. Table 1 collects the four bounds with their validity ranges and closed forms, each derived in Section 2: $\delta_{\mathrm{exact}}$ is the exact closed-form evaluation of $\delta_0$ (a finite Hermite-recurrence sum, Theorem 2); $\delta_{\mathrm{IMF}}$ discards the negative term $-2\mathcal{I}_d^c(\rho x)\, H_{d-1}(\rho x)$ of the Mehta expansion (asymptotically sharp, Theorem 3); $\delta_{\mathrm{SMF}}$ further collapses the Hermite sums to their Szegő monomial envelope, giving an inversion-friendly closed form (Theorem 5); and $\delta_{\mathrm{SM}}$ is an independent Ben Arous–Dembo–Guionnet/layer-cake bound not using the Mehta–Fyodorov algebra (Theorem 6). The first three are strictly nested, $\delta_{\mathrm{exact}} \le \delta_{\mathrm{IMF}} \le \delta_{\mathrm{SMF}}$ on $[u_{\mathrm{IMF}}, \infty)$ (Theorems 2 and 7), and dominate $\delta_{\mathrm{SM}}$ on $[u_{\mathrm{SM}}, \infty)$ (Remark 2). All four decay at the leading rate $u^{d-2} e^{-u^2/2}$ as $u \to \infty$ ($d$ fixed); their $d \to \infty$ prefactors are compared below.

**Explicit form of the constants of Table 1.** The constants originate in the Mehta–Fyodorov representation (2.3): $\alpha_d, \beta_d$ are the dominant Hermite coefficient and remainder envelope in the splitting $Q_d = \alpha_d H_{d-1} + \mathcal{R}_d$ (Lemma 5); $\Phi_d, \Psi_d$ arise in the IMF decomposition (Proposition 2); and $\eta_d$ is the layer-cake correction (Proposition 4). Set

$$c_j \;:=\; (2^j j! \sqrt{\pi})^{-1/2}\,, \qquad \Lambda \;:=\; 2\rho^2 - 1 \;=\; \tfrac{1}{k-1}\,, \qquad \mu_m \;:=\; \int_{\mathbb{R}} H_m(y)\, e^{-y^2/2}\, dy\,;$$

| Method | Validity range | Closed-form expression |
|---|---|---|
| Baseline (not a bound) | $u \to \infty$, $d$ fixed | $\delta_{\mathrm{bl}}(u) \;=\; \dfrac{\sqrt{2}}{\Gamma(d/2)}\left(\dfrac{k}{2}\right)^{\frac{d-1}{2}} u^{d-2}e^{-u^2/2}$ |
| $\delta_{\mathrm{exact}}$ (canonical) | $u \in \mathbb{R}$ (exact) | $\delta_{\mathrm{exact}}(u) \;=\; 2(k-1)^{\frac{d-1}{2}}\big[D_1(u)+D_2(u)+D_3(u)+D_4(u)\big]$, finite Hermite sum (Thm. 2) |
| $\delta_{\mathrm{IMF}}$ | $u \geq u_{\mathrm{IMF}}$ | $\delta_{\mathrm{IMF}}(u) \;=\; 2(k-1)^{\frac{d-1}{2}}\left(\alpha_d\,\Phi_d(\rho,u)\,e^{-u^2/2}+\Psi_d(\rho,u)\,e^{-(1+\rho^2)u^2/2}\right)$ |
| $\delta_{\mathrm{SMF}}$ | $u \geq u_{\mathrm{SMF}}$ | $\delta_{\mathrm{SMF}}(u) \;=\; 4\,\alpha_d(2k)^{\frac{d-1}{2}}u^{d-2}e^{-u^2/2}+2^d\beta_d(k-1)^{\frac{d-1}{2}}u^{2d-3}e^{-3u^2/4}$ |
| $\delta_{\mathrm{SM}}$ | $u \geq u_{\mathrm{SM}}$ | $\delta_{\mathrm{SM}}(u) \;=\; 2\,\dfrac{\sqrt{2}}{\Gamma(d/2)}\left(\dfrac{k}{2}\right)^{\frac{d-1}{2}} u^{d-2}\,e^{-u^2/2}\,(1+\eta_d(\rho,u))$ |

Table 1: The four explicit non-asymptotic tail bounds on $\delta_0(u)$, with the asymptotic baseline $\delta_{\mathrm{bl}}$ (first row), which is the equivalent $\delta_0 \sim \delta_{\mathrm{bl}}$ as $u \to \infty$ ($d$ fixed), not an upper bound. The exact bound $\delta_{\mathrm{exact}}$ (components $D_1, \ldots, D_4$, Theorem 2) coincides with $\delta_0$ for every $u \in \mathbb{R}$; the threshold $u_{\mathrm{IMF}}$ marks only the nesting range $\delta_{\mathrm{exact}} \leq \delta_{\mathrm{IMF}} \leq \delta_{\mathrm{SMF}}$. The constants $\alpha_d, \beta_d$ and functions $\Phi_d, \Psi_d, \eta_d$ are given in (1.6) and Section 2.

by parity, $\mu_{2p+1} = 0$, and direct evaluation yields the closed form $\mu_{2p} = \sqrt{2\pi}\,(2p)!/p!$. With these conventions:

- The dominant Hermite coefficient $\alpha_d$ (i.e. the coefficient of $H_{d-1}(\nu)$ in the Mehta expansion of $Q_d(\nu)$) is

$$\alpha_d \;=\; \begin{cases} \frac{1}{2}\sqrt{d/2}\,c_{d-1}\,c_d\,\mu_d & \text{if } d \text{ is even,} \\ 1/\mu_{d-1} & \text{if } d \text{ is odd.} \end{cases}$$

- The remainder envelope constant $\beta_d$ (from the bound $|\mathcal{R}_d(\nu)| \leq \beta_d(1+\nu^2)^{d-1}e^{-\nu^2/2}$ of (2.13c)) has the explicit form

$$\beta_d \;=\; \underbrace{\sum_{j=0}^{d-1} c_j^2\,2^j\left(\frac{(2j)!}{j!}\right)^2}_{S_d\,:\ \text{squared-Hermite contribution}} \;+\; \sqrt{d/2}\,c_{d-1}\,c_d\,\frac{(2d-2)!}{(d-1)!}\,2^{d-1}\,\widetilde{B}_d\,,$$

where $\widetilde{B}_d$ is the explicit constant

$$\widetilde{B}_d \;=\; \max\!\left(\frac{(2d)!\,2^{d+1}}{d!}\,,\ \frac{(2d)!}{d!}\int_0^{\infty}(1+y)^d\,e^{-y^2/2}\,dy\right),$$

coming from the proof of Lemma 5: it dominates the uniform Hermite-tail envelope constant $E_d$ of (A.3), for which one has $E_d \leq \frac{(2d)!\,2^{d+1}}{d!} \leq \widetilde{B}_d$, so that $|\mathcal{I}_d^c(\nu)| \leq \widetilde{B}_d\,(1+|\nu|)^{d-1}e^{-\nu^2/2}$ holds for every $\nu \geq 0$ (see proof of Lemma 5).

- The polynomial–exponential functions $\Phi_d(\rho,u)$, $\Psi_d(\rho,u)$ entering the IMF decomposition (2.9a) and the SM correction $\eta_d(\rho,u)$ of Proposition 4 read

$$\Phi_d(\rho,u) \;=\; \sum_{j=0}^{\lfloor (d-2)/2\rfloor} 2\rho\,(2\Lambda)^j\,\frac{(d-2)!!}{(d-2j-2)!!}\,(2\rho u)^{d-2j-2} \;+\; \frac{\mathbb{1}_{\{d\ \mathrm{odd}\}}}{u}\,(2\Lambda)^{\frac{d-1}{2}}\,(d-2)!!\,,$$

$$\Psi_d(\rho,u) \;=\; \frac{c_0^2}{(1+\rho^2)\,u} \;+\; \sum_{j=1}^{d-1} c_j^2\,\frac{(2\rho)^{2j}\,u^{2j-1}}{1+\rho^2-(2j-1)/u^2}\,,$$

$$\eta_d(\rho,u) \;=\; \left(1+\left(\tfrac{\sqrt{d-1}}{\rho u}\right)^{1/2}\right)^{d-1} - 1 + (d+1)\,2^{d-1}\,e^{-2\rho u\sqrt{d-1}/9}\,. \tag{1.6a}$$

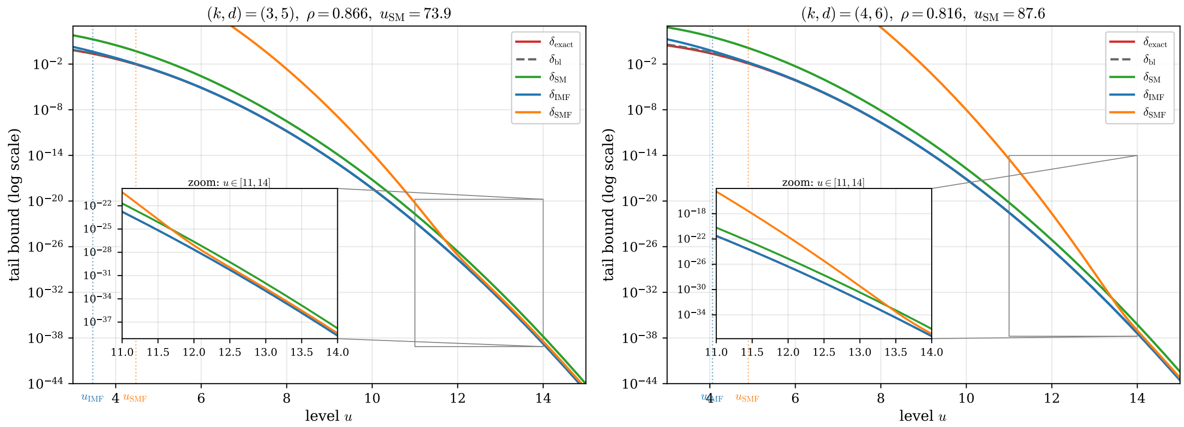


Figure 1: Comparison of the tail bounds $\delta_{\mathrm{exact}}$, $\delta_{\mathrm{IMF}}$, $\delta_{\mathrm{SMF}}$, $\delta_{\mathrm{SM}}$ against the asymptotic baseline $\delta_{\mathrm{bl}}$, on logarithmic scale, plotted against the level $u$ for $(k,d) = (3,5)$ (left, $u_{\mathrm{SM}} = 73.90$) and $(k,d) = (4,6)$ (right, $u_{\mathrm{SM}} = 87.64$), both well outside the displayed range. The five curves are: (i) the strictly exact closed-form bound $\delta_{\mathrm{exact}}(u)$ of Theorem 2 (rigorous, sharpest closed form, and the reference for $\delta_0$ since it is the closed-form evaluation of the Kac–Rice integral (1.4d) on $[u_{\mathrm{IMF}}, \infty)$); (ii) the asymptotically sharp $\delta_{\mathrm{IMF}}(u)$; (iii) the inversion-friendly $\delta_{\mathrm{SMF}}(u)$ (factor 2 above baseline); (iv) the independent spectral-radius bound $\delta_{\mathrm{SM}}(u)$; and (v) the asymptotic baseline $\delta_{\mathrm{bl}}(u)$ of (1.5) (not a bound). All curves start from $u = u_{\mathrm{IMF}}$; the $\delta_{\mathrm{SM}}$ expression is evaluated below its own validity threshold $u_{\mathrm{SM}}$ to make the comparison with $\delta_{\mathrm{SMF}}$ and $\delta_{\mathrm{IMF}}$ visible (it ceases to be a bound below $u_{\mathrm{SM}}$). An inset (top right) zooms onto the moderate-$u$ region where the gap between $\delta_{\mathrm{IMF}}$, $\delta_{\mathrm{exact}}$ and $\delta_{\mathrm{bl}}$ is largest. $\delta_{\mathrm{IMF}}$ is visually indistinguishable from $\delta_{\mathrm{exact}}$, consistent with the asymptotic sharpness of the Hermite-tail analysis; $\delta_{\mathrm{SM}}$ (resp. $\delta_{\mathrm{SMF}}$) is looser through the explicit correction factor $1 + \eta_d(\rho, u)$ (resp. the Szegő envelope coarsening).

**Asymptotic optimality of the prefactors.** As $d \to \infty$ ($k$ fixed), a Stirling approximation shows all three relaxations capture the exponential scale $(ek/d)^{d/2}$ of the Kac–Rice baseline; the IMF bound recovers the exact leading constant, while SM (the absolute-value split at the edge) and SMF (the Szegő envelope) each incur a uniform factor 2. The derivation is given in Appendix C.

**Master tail bound and main theorem.** By direct application of Theorem 3 (the IMF tail bound, Section 2.3) and the two-sided symmetry $\mathbb{P}\{\Gamma_{1,1} > u\} \le 2\,\mathbb{P}\{\sup_{\boldsymbol{\theta}} X(\boldsymbol{\theta}) > u\}$, the master tail bound on $\Gamma_{1,1} = \sup_{\boldsymbol{\theta} \in \mathbb{S}^{d-1}} |X(\boldsymbol{\theta})|$ used in Theorem 1 is

$$\forall u \ge u_{\mathrm{IMF}}\,, \qquad \delta_{\min}(u) \;:=\; 2\,\delta_{\mathrm{IMF}}(u)\,, \qquad \mathbb{P}\big\{\Gamma_{1,1} > u\big\} \;\le\; \delta_{\min}(u)\,. \tag{1.7a}$$

The choice (1.7a) is unconditionally smaller than the SMF branch $2\delta_{\mathrm{SMF}}$ on $[u_{\mathrm{SMF}}, \infty)$ (Theorem 7, Section 2.6), and smaller than the SM branch $2\delta_{\mathrm{SM}}$ on $[u_{\mathrm{SM}}, \infty)$ for every $(k,d)$ of practical interest (Remark 2). The SM and SMF bounds are retained throughout the paper as pedagogical and practical alternatives: $\delta_{\mathrm{SMF}}$ enables the closed-form inversion of Remark 1, $\delta_{\mathrm{SM}}$ is independent of the Mehta–Fyodorov algebra and serves as a consistency check.

**Theorem 1** (Non-asymptotic estimation error bound)**.** *Let $k \ge 3$, $d \ge 3$, $R \ge 1$, let $\boldsymbol{\sigma}^\star \in \mathfrak{S}_R$ be a signal tensor with $\kappa(\boldsymbol{\sigma}^\star) \ge \kappa$, and let $\hat{\boldsymbol{\sigma}}$ be the corresponding profile MLE defined in (1.3). For every $u \ge u_{\mathrm{IMF}} = \sqrt{2d-1}/\rho$, with probability at least $1 - \delta_{\min}(u)$,*

$$\|\hat{\boldsymbol{\sigma}} - \boldsymbol{\sigma}^\star\|_F^2 \;\le\; \frac{4\sqrt{R}\,u}{\kappa\,\lambda}\,. \tag{1.7b}$$

The proof of Theorem 1, given in Section 3.2, combines the deterministic Tube Method (Lemma 8), the rank-reduction inequality (Lemma 9), the symmetry reduction (3.4), and the IMF tail bound (Theorem 3). The choice $\delta_{\min} = 2\,\delta_{\mathrm{IMF}}$ in (1.7a) is unconditionally tighter than $2\,\delta_{\mathrm{SMF}}$ on $[u_{\mathrm{SMF}}, \infty)$ by Theorem 7; the corresponding comparison with $2\,\delta_{\mathrm{SM}}$ on $[u_{\mathrm{SM}}, \infty)$ holds for every $(k,d)$ of practical interest (Remark 2). The strictly tighter substitute $2\,\delta_{\mathrm{exact}}(u) \le \delta_{\min}(u)$ is available for numerical evaluation (Theorem 2).

**Tighter, weaker, and inverted variants** The failure probability $\delta_{\min}(u) = 2\,\delta_{\mathrm{IMF}}(u)$ in Theorem 1 admits a strictly tighter substitute $2\,\delta_{\mathrm{exact}}(u)$ (Theorem 2, Section 2.2) and a more analytically tractable, looser substitute $2\,\delta_{\mathrm{SMF}}(u)$ (Theorem 5, Section 2.4). The choice between them is purely a matter of computational convenience: $\delta_{\mathrm{exact}}$ is preferred when the bound is to be evaluated numerically; $\delta_{\mathrm{SMF}}$ when an explicit closed-form threshold $u_\alpha$ is required (Remark 1 and Appendix B); $\delta_{\mathrm{IMF}}$ is the natural default. The independent $\delta_{\mathrm{SM}}$ (Theorem 6, Section 2.5) is dominated by $\delta_{\mathrm{IMF}}$ on $[u_{\mathrm{SM}}, \infty)$ for every $(k, d)$ of practical interest (Remark 2) and is retained for its conceptual independence from the Mehta–Fyodorov algebra and as an unconditional check; in all cases the conservative bound $\min(2\delta_{\mathrm{IMF}}, 2\delta_{\mathrm{SM}})$ is unconditional on $[u_{\mathrm{SM}}, \infty)$.

**Remark 1** (Choice of the level $u$)**.** *For a target confidence level $\alpha \in (0, 1)$, set*

$$u \;=\; \sqrt{2\log(1/\alpha) \,+\, n\log(2k) \,+\, 2n\log\log(1/\alpha)}\,, \qquad n = d-1. \tag{1.8}$$

*There exists an explicit threshold $\alpha_0(k, d) \in (0, 1)$ such that the choice (1.8) satisfies $\delta_{\min}(u) \le \alpha$ for every $\alpha \in (0, \alpha_0(k, d)]$. The dependence on $(k, d)$ enters only through the constants $\alpha_d$, $\beta_d$ of Lemma 5 and $u_d^\star$ (the level beyond which the SMF remainder is negligible; Corollary 3), together with the absolute constant of the Gaussian tail bound; no further hidden quantities are involved. This threshold is exponentially small in the dimension: the closed form (1.8) is calibrated against the SMF main term, itself a bound only on $[u_{\mathrm{SMF}}, \infty)$ with $u_{\mathrm{SMF}} = 2\sqrt{d}$, so $\alpha_0(k, d) = O(e^{-2d})$ up to a polynomial-in-$d$ prefactor and the SMF bound is vacuous ($> 1$) over the moderate-$\alpha$ range. For moderate confidence levels one therefore inverts the non-vacuous master bound $\delta_{\min} = 2\delta_{\mathrm{IMF}}$ of Theorem 1 numerically; this affects only the availability of the closed form (1.8), not the validity of Theorem 1. Substituting (1.8) into (1.7b) yields the non-asymptotic rate*

$$\|\hat{\boldsymbol{\sigma}} - \boldsymbol{\sigma}^\star\|_F^2 \;\lesssim\; \frac{\sqrt{R}}{\kappa\,\lambda}\,\sqrt{d\log k - \log\alpha}\,,$$

*which recovers, with explicit constants and an explicit dependence on $(R, \kappa)$, the asymptotic optimal rate of Perry et al. (2020) established for $d \to \infty$ in the rank-one case. The full proof, given in Appendix B, performs the inversion term-by-term against the SMF main contribution and absorbs the polynomial $\log u$ correction arising from $u^{n-1}$ into the $2n\log\log(1/\alpha)$ slack of (1.8).*

## 1.4 Related work and contributions

Tensor methods are widely used for learning latent variable models: community detection in stochastic block models, parameter estimation in mixtures of Gaussians, latent Dirichlet allocation, and Independent Component Analysis can all be recast as the low-rank decomposition of an empirical moment tensor of order $k \ge 3$, which resolves the identifiability obstructions that, for the matrix case $k = 2$, prevent unique recovery of the low-rank components from the moment tensor alone (Anandkumar et al., 2014). Spiked Tensor PCA, introduced by Richard and Montanari (2014), isolates the low-rank-plus-noise structure of the estimation problem, abstracting away problem-specific details. Perry et al. (2020, Theorem 1.3) establish that detecting a rank-one signal below an explicit eigenvalue threshold is information-theoretically impossible as $d \to \infty$, and identify the optimal estimation rate $O(\sqrt{d\log k})$ in our normalization (which differs from theirs by a factor of $\sqrt{d/2}$, the noise-normalisation conversion detailed in Azaïs et al., 2024). Theorem 1 recovers this rate non-asymptotically, with explicit constants and an explicit dependence on the rank $R$ and the coherence $\kappa$ that does not appear in the existing asymptotic literature (see Remark 1). The matrix-analogue spectral signature of the detection threshold is the Baik–Ben Arous–Péché (BBP) phase transition (Baik et al., 2005), describing how the extreme eigenvalues of a deformed random matrix detach from the bulk only once the signal-to-noise ratio crosses a precise threshold; this transition is closely connected to the spectral edge of the Gaussian Orthogonal Ensemble whose conditional law is used throughout our Kac–Rice analysis.

The geometric landscape of the likelihood function for tensor PCA is a smooth random function on the high-dimensional sphere, and its critical-point structure has been studied in the spin-glass literature. Auffinger et al. (2013) establish that the number of critical points of the same Kostlan–Shub–Smale field vanishes above the spectral edge and relate this complexity to GOE matrix theory; we exploit in particular their conditional Hessian law, Auffinger et al. (2013, Lemma 3.2(b)), as a main input to the Kac–Rice computation. The present paper differs from this body of work in focus: rather than counting critical points, we control the tail of the global maximum, and our bounds can be read as a non-asymptotic refinement of the large-deviation complexity theory in the region above the spectral edge. The connection to spin glasses extends further: aging phenomena and energy-landscape complexity in the $p$-spin spherical glass model studied by Ben Arous et al. (2001), Fyodorov (2004) and Ben Arous et al. (2019) are direct probabilistic analogues of the optimization problem we analyze, and the topology of local maxima governs the detectability threshold in both settings. On the algorithmic side, the high-dimensional dynamics of efficient solvers for the multi-spiked tensor model (Langevin dynamics and online stochastic gradient descent) were recently characterised by Ben Arous et al. (2024a,b), who determine the sample complexity and signal-to-noise separation required for recovery; these address the *computational* side of the statistical-to-computational gap, complementary to the *statistical* profile-MLE guarantee of Theorem 1.

The canonical Kostlan–Shub–Smale random field, originally introduced by Shub and Smale (1993) and Kostlan (1993) to study the distribution of roots of random polynomial systems, is at the algebraic core of our analysis. Its covariance structure $\langle \boldsymbol{\theta}, \boldsymbol{v} \rangle^k$ matches that of our tensor noise exactly, and underlies the structured conditioning of field value and Hessian. Our methodology is the Kac–Rice framework for level sets and extrema of random fields, treated in detail by Azaïs and Wschebor (2009). The present paper is a companion to Azaïs et al. (2024), where the same conditional Kac–Rice analysis of the Kostlan–Shub–Smale field underlies an exact *spacing test* for detecting sparse alternatives in Gaussian symmetric tensors, including an exact, non-asymptotic test for the spiked tensor model that needs no prior knowledge of the noise level, thus providing the *detection* counterpart to the estimation and complexity results obtained here. Sharpening the resulting integrals depends on orthogonal-polynomial expansions of the GOE eigenvalue density formalised in Mehta (2004) and on Szegő's bounds on Hermite polynomial roots (Szegő, 1975).

**Contributions.** *(i)* A geometric framework for rank-$R$ tensor estimation via the profile MLE on $\mathfrak{S}_R$, with a deterministic Tube Method bound and a rank-reduction inequality controlling the rank-$R$ noise level by the rank-one Kostlan–Shub–Smale supremum (Lemmas 8, 9). *(ii)* A four-tier hierarchy of explicit non-asymptotic tail bounds on that supremum: $\delta_{\text{exact}}$ (Theorem 2), the asymptotically sharp $\delta_{\text{IMF}}$ (Theorem 3), the inversion-friendly $\delta_{\text{SMF}}$ (Theorem 5), and the independent $\delta_{\text{SM}}$ (Theorem 6), with the domination established in Theorem 7. *(iii)* A unified main theorem (Theorem 1) with explicit failure probability $\delta_{\min} = 2\delta_{\text{IMF}}$, recovering the Perry–Wein–Bandeira rate $\sqrt{d \log k}$ with explicit $(R, \kappa)$-dependence. *(iv)* A non-asymptotic two-sided bracketing of the annealed complexity of the spherical $k$-spin Hamiltonian (Theorem 8): writing $N^{\text{lm}}_{[E,\infty)}$ for the number of local maxima of $X$ with $X(\boldsymbol{\theta}) \geq E$, for every finite $(k, d)$ and every $E \geq E_{\text{BDG}} = 8\sqrt{2(d-1)}/\rho$,

$$\left(1 - C_{\text{amp}}^{d-1} \delta_{\text{BDG}}^{1/2}(\rho E)\right) \delta_{\text{exact}}(E) \;\leq\; \mathbb{E}\left[N^{\text{lm}}_{[E,\infty)}\right] \;\leq\; \delta_{\text{exact}}(E)\,,$$

where $\delta_{\text{BDG}}(\rho E) = e^{-2(\rho E)^2/9}$, $C_{\text{amp}} = 8\sqrt{2}$, and the factor $C_{\text{amp}}^{d-1}\delta_{\text{BDG}}^{1/2}$ vanishes super-exponentially in $d$. This recovers the Auffinger–Ben Arous–Černý complexity function (Auffinger et al., 2013, Theorem 2.4) as $d \to \infty$ (Corollary 9, with the dual very-deep-minima statement in Corollary 8; at reduced energy $e = \rho E/\sqrt{2(d-1)} \geq 1$, the finite-$d$ bracket holding for $e \geq 8$), and connects to the spiked landscape of Ben Arous et al. (2019).

## 1.5 Notation

A comprehensive table of all mathematical notations, grouped by topic, is provided in Appendix E; we record here only the conventions used most frequently in the body. We denote by $\mathcal{T}(k, d)$ the space of symmetric tensors of order $k$ and dimension $d$, equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{T}}$ and the Frobenius norm $\| \cdot \|_F$; $\mathbb{S}(k, d)$ is the unit sphere in $\mathcal{T}(k, d)$. The unit sphere of $\mathbb{R}^d$ is $\mathbb{S}^{d-1}$, with surface area $|\mathbb{S}^{d-1}| = 2\pi^{d/2}/\Gamma(d/2)$. The subset $\mathfrak{S}_R \subset \mathbb{S}(k, d)$ collects tensors of rank at most $R$. The standard Gaussian tail is $\bar{\Phi}(u) := \mathbb{P}(Z > u)$ with $Z \sim \mathcal{N}(0, 1)$, and $\varphi(z) := (2\pi)^{-1/2} e^{-z^2/2}$. The integer $n := d - 1$ denotes the size of the GOE matrices arising in the Kac–Rice analysis (dimension of the tangent space to $\mathbb{S}^{d-1}$); we use $n$ exclusively for the GOE matrix size and $d$ for the ambient tensor dimension.

## 2 Expected GOE characteristic polynomial and Kac–Rice tail bounds

This section assembles a hierarchy of four explicit non-asymptotic bounds on the Kac–Rice integral (1.4d). Throughout, the object of interest is the supremum probability $\mathbb{P}\{\sup_{\boldsymbol{\theta}\in\mathbb{S}^{d-1}} X(\boldsymbol{\theta}) > u\}$; the passage to $\mathbb{P}\{\Gamma_{1,1} > u\}$ relevant for Theorem 1 is the symmetry reduction (3.4) of Section 3.2, applied once at the end of the argument. The Kac–Rice formula (Azaïs and Wschebor, 2009, Theorem 6.4) bounds this supremum probability by an integral against the intensity of *critical points* of $X$ at level $x$, which by isotropy reduces to the Gaussian integral against the expected absolute determinant of the conditional Hessian, namely

$$\mathbb{P}\Big\{\sup_{\boldsymbol{\theta}\in\mathbb{S}^{d-1}} X(\boldsymbol{\theta}) > u\Big\} \le \int_u^\infty \bar{p}(x)\,\mathrm{d}x\,, \qquad \bar{p}(x) := |\mathbb{S}^{d-1}|\, p_{\nabla X}(0)\,\mathbb{E}\big[|\det \nabla^2 X(\boldsymbol{\theta})|\,\big|\, X(\boldsymbol{\theta}) = x\big]\varphi(x)\,,$$

where $\bar{p}(x)$ is the intensity of critical points of $X$ at level $x$; $p_{\nabla X}(0) = (2\pi k)^{-\frac{d-1}{2}}$; and for any fixed $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$, $\nabla^2 X(\boldsymbol{\theta})$ denotes the Riemannian Hessian at $\boldsymbol{\theta}$. The bound

$$\mathbb{P}\Big\{\sup_{\boldsymbol{\theta}\in\mathbb{S}^{d-1}} X(\boldsymbol{\theta}) > u\Big\} \le \mathbb{E}\Big[\#\big\{\boldsymbol{\theta}\in\mathbb{S}^{d-1} : \nabla X(\boldsymbol{\theta}) = 0,\ X(\boldsymbol{\theta}) > u\big\}\Big] = \int_u^\infty \bar{p}(x)\,\mathrm{d}x\,,$$

holds because, on the event $\{\sup_{\boldsymbol{\theta}} X(\boldsymbol{\theta}) > u\}$, the smooth field $X$ attains its supremum at an interior critical point of the compact manifold $\mathbb{S}^{d-1}$, so the indicator of this event is dominated by the count of critical points above $u$. The bound is loose only by the (positive) contribution of saddle points above $u$; this slack is asymptotically negligible above the spectral edge and is the cost of a closed-form Kac–Rice integrand. The conditioning on the level $X(\boldsymbol{\theta}) = x$ alone (rather than jointly on $\{\nabla X(\boldsymbol{\theta}) = 0,\ X(\boldsymbol{\theta}) = x\}$, as the Kac–Rice intensity formally requires) is justified by isotropy: at any fixed $\boldsymbol{\theta}$ the cross-covariances $\mathbb{E}[X(\boldsymbol{\theta})\,\partial_i X(\boldsymbol{\theta})]$ and $\mathbb{E}[\partial_i X(\boldsymbol{\theta})\,\nabla^2 X(\boldsymbol{\theta})]$ vanish, so the Gaussian gradient $\nabla X(\boldsymbol{\theta})$ is independent of the pair $(X(\boldsymbol{\theta}), \nabla^2 X(\boldsymbol{\theta}))$ and the event $\{\nabla X(\boldsymbol{\theta}) = 0\}$ may be dropped from the conditioning of the determinant. The conditional Hessian is itself, by Auffinger et al. (2013, Lemma 3.2(b)), a shifted GOE matrix:

$$\mathbb{E}\big[|\det \nabla^2 X(\boldsymbol{\theta})|\,\big|\, X(\boldsymbol{\theta}) = x\big] \;=\; (2k(k-1))^{\frac{d-1}{2}}\,\mathbb{E}\big[|\det(G_{d-1} - \rho x\, I_{d-1})|\big]\,, \tag{2.1a}$$

and combining these inputs yields the central reduction

$$\mathbb{P}\Big\{\sup_{\boldsymbol{\theta}\in\mathbb{S}^{d-1}} X(\boldsymbol{\theta}) > u\Big\} \;\le\; C_{k,d}\int_u^\infty \mathbb{E}\big[|\det(G_{d-1} - \rho x\, I_{d-1})|\big]\,\varphi(x)\,\mathrm{d}x\,, \tag{2.1b}$$

where $\rho$ and $C_{k,d}$ are given in (1.4f). We develop, in turn, the four explicit closed-form bounds on the right-hand side.

The remainder of this section derives the bounds, tightest first, with the sharpened variant $\delta^\star_{\mathrm{IMF}}$ interpolating between $\delta_{\mathrm{exact}}$ and $\delta_{\mathrm{IMF}}$: they are nested as $\delta_{\mathrm{exact}} \le \delta^\star_{\mathrm{IMF}} \le \delta_{\mathrm{IMF}} \le \delta_{\mathrm{SMF}}$ on $[u_{\mathrm{IMF}}, \infty)$ (Theorems 2, 4, 7), with the independent $\delta_{\mathrm{SM}}$ dominated on $[u_{\mathrm{SM}}, \infty)$ (Remark 2); $\delta_{\mathrm{IMF}}$ is the bound used in Theorem 1.

### 2.1 Mehta–Fyodorov representation (MF)

The Mehta–Fyodorov representation uses two identities: Fyodorov's formula (Lemma 1), which writes the expected absolute characteristic polynomial of a $(d-1)$-GOE at level $\nu$ through the one-point eigenvalue density of a one-size-larger $d$-GOE, and Mehta's Hermite expansion of that density (Lemma 2); together they give an exact closed algebraic form of the Kac–Rice integrand.

**Lemma 1** (Fyodorov's formula (Fyodorov, 2004))**.** *For every $\nu \in \mathbb{R}$,*

$$\mathbb{E}\big[|\det(G_{d-1} - \nu I_{d-1})|\big] \;=\; \frac{2^{\frac{3}{2}}\,\Gamma((d+2)/2)}{d}\,Q_d(\nu)\,, \tag{2.2}$$

*where $Q_d(\nu) := e^{\nu^2/2} q_d(\nu)$ and $q_d$ is the eigenvalue intensity (one-point correlation function) of a $d \times d$ GOE matrix in the Mehta normalization, so that $\int_{\mathbb{R}} q_d(\nu)\,\mathrm{d}\nu = d$, the expected number of eigenvalues. The identity expresses the expected absolute characteristic polynomial of the $(d-1)\times(d-1)$ GOE through the one-point function of the* one-size-larger *$d \times d$ GOE; see Fyodorov (2004).*

**Lemma 2** (Mehta expansion (Mehta, 2004, p. 221)). *The GOE one-point function admits the following Hermite-function expansion. With* $c_j := (2^j j! \sqrt{\pi})^{-1/2}$ *and* $H_j$ *the j-th physicist Hermite polynomial,*

$$Q_d(\nu) \;=\; \frac{1}{2}\sqrt{\frac{d}{2}}\, c_{d-1} c_d\, H_{d-1}(\nu)\big[\mu_d - 2\mathcal{I}_d^c(\nu)\big] \;+\; \mathbb{1}_{\{d \text{ odd}\}}\, \frac{H_{d-1}(\nu)}{\mu_{d-1}} + e^{-\nu^2/2} \sum_{j=0}^{d-1} c_j^2\, H_j^2(\nu)\,, \tag{2.3}$$

*where*

$$\mu_m := \int_{\mathbb{R}} H_m(y)\, e^{-y^2/2}\, \mathrm{d}y\,, \qquad \mathcal{I}_d^c(\nu) := \int_\nu^\infty H_d(y)\, e^{-y^2/2}\, \mathrm{d}y\,.$$

*By parity,* $\mu_m = 0$ *whenever m is odd; in particular, for d odd only the* $H_{d-1}/\mu_{d-1}$ *term contributes, whereas for d even only the* $\mu_d$ *bracket does.*

The combination of Lemmas 1 and 2 produces an exact integral representation of the Kac–Rice probability bound. Fyodorov's formula (2.2) gives the expected absolute characteristic polynomial as a constant multiple of $Q_d(\rho x)$, and Mehta's expansion (2.3) supplies an explicit formula for $Q_d$. Substituting into the Kac–Rice reduction (2.1b) therefore yields an integrand of the form $Q_d(\rho x)\, e^{-x^2/2}$: a polynomial-and-Hermite quantity weighted by the original Kac–Rice Gaussian, with no further exponential factor entering the integrand. Fyodorov's identity is already stated in the natural variable $Q_d$, so the eigenvalue intensity $q_d = e^{-\nu^2/2} Q_d$ never appears in the substitution: the Gaussian envelope that would otherwise be carried by $q_d$ is absorbed into the definition of $Q_d$. This algebraic identity, unavailable to the spectral-splitting route, is why the IMF and SMF bounds of Sections 2.3 and 2.4 close the residual factor-2 gap of Theorem 6. We state this in the following proposition.

**Proposition 1** (Exact MF representation). *For every* $u \in \mathbb{R}$,

$$\mathbb{P}\Big\{ \sup_{\boldsymbol{\theta} \in \mathbb{S}^{d-1}} X(\boldsymbol{\theta}) > u \Big\} \;\le\; 2(k-1)^{\frac{d-1}{2}} \int_u^\infty Q_d(\rho x)\, e^{-x^2/2}\, \mathrm{d}x\,. \tag{2.4}$$

*Proof of Proposition 1.* • Starting from the Kac–Rice reduction in (2.1b), Lemma 1 evaluated at the shifted argument $\nu = \rho x$ gives an exact expression for the expected absolute determinant directly in terms of $Q_d$:

$$\mathbb{E}\big[|\det(G_{d-1} - \rho x\, I_{d-1})|\big] \;=\; \frac{2^{\frac{3}{2}} \Gamma((d+2)/2)}{d}\, Q_d(\rho x)\,, \tag{2.5}$$

where $Q_d(\nu) = e^{\nu^2/2} q_d(\nu)$ is the unweighted polynomial part of the eigenvalue intensity (one-point correlation function) of a $d \times d$ GOE matrix in the Mehta normalization. No exponential factor in $\rho x$ is introduced at this step: the Gaussian envelope that would otherwise be carried by $q_d$ is precisely what the definition of $Q_d$ absorbs.
• We multiply this expression by the standard Gaussian density $\varphi(x) = (2\pi)^{-1/2} e^{-x^2/2}$ and integrate over the interval $[u, \infty)$ to match the Kac–Rice integral:

$$\int_u^\infty \mathbb{E}\big[|\det(G_{d-1} - \rho x\, I_{d-1})|\big]\, \varphi(x)\, \mathrm{d}x \;=\; \frac{2^{\frac{3}{2}} \Gamma((d+2)/2)}{d\sqrt{2\pi}} \int_u^\infty Q_d(\rho x)\, e^{-x^2/2}\, \mathrm{d}x\,.$$

We can simplify the constant prefactor using the Gamma function identity $\Gamma((d+2)/2) = (d/2)\Gamma(d/2)$:

$$\frac{2^{\frac{3}{2}} \Gamma((d+2)/2)}{d\sqrt{2\pi}} \;=\; \frac{2\sqrt{2} \cdot (d/2)\Gamma(d/2)}{d\sqrt{2}\sqrt{\pi}} \;=\; \frac{\Gamma(d/2)}{\sqrt{\pi}}\,.$$

Finally, substituting this result back into the original Kac–Rice bound (2.1b) alongside the geometric volume prefactor $C_{k,d} = 2\sqrt{\pi}(k-1)^{\frac{d-1}{2}}/\Gamma(d/2)$, the $\Gamma(d/2)$ and $\sqrt{\pi}$ terms cancel:

$$C_{k,d} \frac{\Gamma(d/2)}{\sqrt{\pi}} \;=\; \left( \frac{2\sqrt{\pi}(k-1)^{\frac{d-1}{2}}}{\Gamma(d/2)} \right) \frac{\Gamma(d/2)}{\sqrt{\pi}} \;=\; 2(k-1)^{\frac{d-1}{2}}\,.$$

This yields the exact representation (2.4). □

The identity (2.4) is an *exact representation*, not yet a bound: it expresses the Kac–Rice probability as an explicit integral against the Mehta function $Q_d(\rho x)$, which itself comprises a linear Hermite term $H_{d-1}(\rho x)$, a squared Hermite sum $\sum_j c_j^2 H_j^2(\rho x)$, and a signed tail integral $\mathcal{I}_d^c(\rho x)$. The IMF (Section 2.3) and SMF (Section 2.4) bounds both take this identity as their starting point; they differ only in how aggressively they coarsen these three components. The IMF bound preserves the full partial Hermite sum structure and is asymptotically sharp; the SMF bound trades a small constant loss for an explicit, closed-form polynomial–exponential envelope.

## 2.2 Strictly exact Mehta–Fyodorov tail bound

Substituting the full Mehta expansion (2.3) into Proposition 1 produces the four-piece decomposition

$$\int_u^\infty Q_d(\rho x)\, e^{-x^2/2}\, \mathrm{d}x \;=\; D_1(u) \;+\; D_2(u) \;+\; D_3(u) \;+\; D_4(u)\,, \tag{2.6}$$

where, with $\nu = \rho x$,

$$\begin{aligned}
D_1(u) \;&:=\; \tfrac{1}{2}\sqrt{d/2}\, c_{d-1} c_d\, \mu_d\, \mathcal{L}_d(u)\,, && \text{(zero for } d \text{ odd, since } \mu_d = 0\text{)},\\
D_2(u) \;&:=\; -\sqrt{d/2}\, c_{d-1} c_d\, \mathcal{C}_d(u)\,, && \text{(cross integral } \mathcal{C}_d \text{ defined in (2.7))},\\
D_3(u) \;&:=\; \mathbb{1}_{\{d\,\text{odd}\}}\, \mu_{d-1}^{-1}\, \mathcal{L}_d(u)\,, && \text{(zero for } d \text{ even)},\\
D_4(u) \;&:=\; T_d^{\text{exact}}(u) && \text{(squared-Hermite tail of (2.12))},
\end{aligned}$$

with $\mathcal{L}_d(u) := \int_u^\infty H_{d-1}(\rho x)\, e^{-x^2/2}\, \mathrm{d}x$ and the cross integral

$$\mathcal{C}_d(u) \;:=\; \int_u^\infty H_{d-1}(\rho x)\, \mathcal{I}_d^c(\rho x)\, e^{-x^2/2}\, \mathrm{d}x\,. \tag{2.7}$$

**Theorem 2** (Strictly exact MF tail bound)**.** *For every $u \in \mathbb{R}$,*

$$\mathbb{P}\Big\{ \sup_{\boldsymbol{\theta} \in \mathbb{S}^{d-1}} X(\boldsymbol{\theta}) > u \Big\} \;\le\; \delta_{\text{exact}}(u) \;:=\; 2(k-1)^{\frac{d-1}{2}} \big[ D_1(u) + D_2(u) + D_3(u) + D_4(u) \big]\,, \tag{2.8a}$$

*where each $D_i$ is given in fully closed form by:*

($D_1, D_3$) *the linear Hermite tail $\mathcal{L}_d(u)$ admits the closed form*

$$\mathcal{L}_d(u) \;=\; \frac{1}{\rho}\, J_{d-1}^{1/\rho^2}(\rho u)\,, \tag{2.8b}$$

*with $J_m^\beta$ given by Lemma 15;*

($D_4$) *the squared Hermite tail $T_d^{\text{exact}}(u) = \rho^{-1} \sum_{j=0}^{d-1} c_j^2\, K_j^\beta(\rho u)$ of (2.12), with $\beta = (3k-2)/k$ and $K_j^\beta$ given by Corollary 11;*

($D_2$) *the cross integral $\mathcal{C}_d(u)$ admits the two-piece closed form*

$$\begin{aligned}
\mathcal{C}_d(u) \;=\; &\frac{1}{\rho} \sum_{\ell=0}^{\lfloor (d-1)/2 \rfloor} \frac{2^{\ell+1}\,(d-1)!!}{(d-2\ell-1)!!} \sum_{p=0}^{d-2\ell-1} 2^p\, p! \binom{d-1}{p} \binom{d-2\ell-1}{p} J_{2d-2\ell-2-2p}^\beta(\rho u)\\
&+\; \mathbb{1}_{\{d\,\text{even}\}}\, 2^{d/2}\,(d-1)!!\, \mathcal{F}_d(u)\,,
\end{aligned} \tag{2.8c}$$

*where the Fubini remainder $\mathcal{F}_d(u)$ (only present for $d$ even) reads*

$$\mathcal{F}_d(u) \;=\; \sqrt{2\pi}\, \bar{\Phi}(\rho u)\, \frac{J_{d-1}^{1/\rho^2}(\rho u)}{\rho} \;-\; 2\rho \sum_{\ell=0}^{(d-2)/2} (2\Lambda)^\ell\, \frac{(d-2)!!}{(d-2\ell-2)!!}\, J_{d-2\ell-2}^\beta(\rho u)\,. \tag{2.8d}$$

*Moreover,*

$$\delta_{\text{exact}}(u) \;\le\; \delta_{\text{IMF}}^\star(u) \;\le\; \delta_{\text{IMF}}(u) \qquad \textit{for every } u \ge u_{\text{IMF}},$$

*with strict inequality at every finite $u$.*

$\delta_{\text{exact}}$ has two roles. The quantity $2(k-1)^{(d-1)/2}\sum_{i=1}^{4} D_i(u)$ is the closed-form *evaluation* of the Kac–Rice integral $2(k-1)^{(d-1)/2}\int_u^\infty Q_d(\rho x)e^{-x^2/2}\,dx$ of Proposition 1; by the same Kac–Rice identity it equals, *exactly*, the expected number of critical points of $X$ above level $u$ (cf. Remark 6 and Section 4). The inequality (2.8a) for $\mathbb{P}\{\sup X > u\}$ is then the first-moment (Markov) step

$$\mathbb{P}\{\sup X > u\} \;\leq\; \mathbb{E}\big[\#\{\text{critical points above } u\}\big]\,.$$

Thus $\delta_{\text{exact}}$ is an equality for the critical-point count and an upper bound for the supremum probability; both readings are used in Sections 3 and 4.

*Proof.* Deferred to Appendix A. □

The identity $\delta_{\text{exact}}(u) = 2(k-1)^{(d-1)/2}\sum_{i=1}^{4} D_i(u)$ matches high-precision quadrature of the integral (2.4) to relative error $10^{-13}$–$10^{-8}$ on the grid $(k,d) \in \{(3,3),(3,5),(4,6),(5,7),(3,10)\}$; the full symbolic and numerical certification is in the companion repository `github.com/ydecastro/TENSOR-KSS`.

## 2.3 Improved Mehta–Fyodorov bound (IMF)

Concretely, inspecting the Mehta expansion (2.3) one sees that the coefficient of the Hermite polynomial $H_{d-1}(\nu)$ is the bracket $[\mu_d - 2\mathcal{I}_d^c(\nu)]$, where $\mathcal{I}_d^c(\nu) := \int_\nu^\infty H_d(y)e^{-y^2/2}\,dy$ is a Hermite tail integral. If a regime can be identified in which $\mathcal{I}_d^c(\nu)$ is *strictly positive*, the subtractive contribution $-2\mathcal{I}_d^c(\nu)H_{d-1}(\nu)$ to $Q_d$ is strictly negative (since $H_{d-1}(\nu) > 0$ in the same regime), and discarding it yields a strict upper bound on the Kac–Rice integral.

Pinpointing this regime reduces to locating the largest root of $H_{d-1}$: beyond it, $H_{d-1}(\nu)$ is positive, and by a root-interlacing argument $\mathcal{I}_d^c(\nu)$ is too. Hence the threshold is set by the largest root of $H_{d-1}$. Szegő's classical estimate (Szegő, 1975, Theorem 6.32) locates this largest root strictly below $\sqrt{2d-1}$, so that the relevant regime is $\nu \geq \sqrt{2d-1}$, which translates into the threshold $\rho u \geq \sqrt{2d-1}$ in the Kac–Rice integral. This threshold is smaller than the Spectral Method threshold $32\sqrt{d-1}/\rho$ and incurs no corrective factor $1+\eta_d(\rho,u)$.

The remainder of this subsection assembles the three ingredients required to turn the discarding argument into an explicit bound: a positivity-and-envelope statement for the Hermite tail integral $\mathcal{I}_d^c$ (Lemma 3); a recurrence-based evaluation of the resulting linear and squared Hermite tail integrals; and the explicit polynomial-rational functions $\Phi_d(\rho,u)$ and $\Psi_d(\rho,u)$ that encode, respectively, the linear Hermite tail and the squared Hermite tail.

**Lemma 3** (Positivity and upper bound of the Hermite tail integral)**.** *Let $d \geq 3$ and let $x_1^{(m)}$ denote the largest root of the (physicist) Hermite polynomial $H_m$. For every $\nu \geq x_1^{(d-1)}$, the tail integral $\mathcal{I}_d^c(\nu)$ is strictly positive. Moreover there exists a dimension-dependent constant $C_d > 0$ such that, for all such $\nu$,*

$$0 \;<\; \mathcal{I}_d^c(\nu) \;\leq\; C_d\,\nu^{d-1}\,e^{-\nu^2/2}\,.$$

*The assumption $d \geq 3$ ensures $x_1^{(d-1)} \geq x_1^{(2)} = 1/\sqrt{2} > 0$, which is used implicitly in the polynomial-envelope step of the proof.*

*Proof of Lemma 3.* • We apply the explicit finite series expansion from Lemma 14 (specifically Equation (D.7a)) with degree $m = d$. This yields:

$$\mathcal{I}_d^c(\nu) = e^{-\nu^2/2}\sum_{k=0}^{\lfloor (d-1)/2\rfloor} 2^{k+1}\frac{(d-1)!!}{(d-2k-1)!!}H_{d-2k-1}(\nu) + R_d(\nu)\,,$$

where the remainder term guarantees $R_d(\nu) \geq 0$.
• The roots of consecutive orthogonal polynomials interlace (Szegő, 1975, Theorem 3.3.2); in particular, the largest root $x_1^{(m)}$ of $H_m$ is strictly increasing in the degree $m$. Therefore, for every $k \geq 1$, we have the strict inequality:

$$x_1^{(d-2k-1)} < x_1^{(d-1)}\,.$$

When we evaluate the sum at any $\nu \geq x_1^{(d-1)}$, the leading polynomial ($k=0$) satisfies $H_{d-1}(\nu) \geq 0$. For all subsequent lower-degree polynomials ($k \geq 1$), since $\nu \geq x_1^{(d-1)} > x_1^{(d-2k-1)}$, they evaluate to strictly positive values: $H_{d-2k-1}(\nu) > 0$. Because all the constant combinatorial coefficients in the sum are strictly positive and $R_d(\nu) \geq 0$, the entire expression evaluates to a strictly positive number: $\mathcal{I}_d^c(\nu) > 0$.

• To upper bound the integral for $\nu \geq x_1^{(d-1)}$, we apply the polynomial envelope from Lemma 4, which bounds each polynomial by $H_{d-2k-1}(\nu) \leq (2\nu)^{d-2k-1}$. The dominant term in the sum corresponds to $k=0$, which is bounded by:

$$2H_{d-1}(\nu)e^{-\nu^2/2} \;\leq\; 2(2\nu)^{d-1}e^{-\nu^2/2}\,.$$

For the lower-order terms ($k \geq 1$) the same envelope gives $H_{d-2k-1}(\nu) \leq (2\nu)^{d-2k-1}$, and we promote the asymptotic gain into a uniform pointwise bound exactly as for the remainder term $R_d$: writing $\nu^{d-2k-1} = \nu^{d-1}\,\nu^{-2k}$, the factor $\nu^{-2k}$ is decreasing on $[x_1^{(d-1)}, \infty)$ and attains its supremum $\big(x_1^{(d-1)}\big)^{-2k} < \infty$ at the left endpoint (here $x_1^{(d-1)} > 0$ is used). Hence each lower-order term is bounded by a constant multiple of the leading envelope $\nu^{d-1}e^{-\nu^2/2}$ on the whole interval $[x_1^{(d-1)}, \infty)$.

It remains to control the remainder $R_d(\nu)$, which is non-zero only when $d$ is even. In that case, Lemma 14 gives the explicit Gaussian-tail expression

$$R_d(\nu) \;=\; 2^{d/2}(d-1)!!\int_\nu^\infty e^{-x^2/2}\,\mathrm{d}x\,.$$

Applying Mills' ratio (equivalently, Lemma 10 with $m=0$, $a=1$, i.e. (D.1a)) yields $\int_\nu^\infty e^{-x^2/2}\,\mathrm{d}x \leq \nu^{-1}e^{-\nu^2/2}$ for $\nu > 0$, whence

$$R_d(\nu) \;\leq\; 2^{d/2}(d-1)!!\,\nu^{-1}\,e^{-\nu^2/2} \;=\; O\big(\nu^{-1}\,e^{-\nu^2/2}\big)\,.$$

To promote this asymptotic decay into a uniform pointwise bound on the full domain: the ratio of the remainder envelope to the leading envelope is $\nu^{-1}/\nu^{d-1} = \nu^{-d}$. Since $x_1^{(d-1)} > 0$, this ratio is a positive continuous function on $[x_1^{(d-1)}, \infty)$ which attains its supremum at the left endpoint:

$$\sup_{\nu \geq x_1^{(d-1)}} \nu^{-d} \;=\; \big(x_1^{(d-1)}\big)^{-d} \;<\; \infty\,.$$

Consequently $R_d(\nu) \leq 2^{d/2}(d-1)!!\,\big(x_1^{(d-1)}\big)^{-d}\,\nu^{d-1}\,e^{-\nu^2/2}$ for every $\nu \geq x_1^{(d-1)}$, a uniform pointwise bound on the whole semi-infinite interval (not only for large $\nu$). Absorbing the leading constant, the lower-order polynomial terms, and the remainder $R_d(\nu)$ into a single dimension-dependent constant $C_d$ yields the final bound:

$$\mathcal{I}_d^c(\nu) \;\leq\; C_d\,\nu^{d-1}e^{-\nu^2/2}\,,$$

□

With the positivity of $\mathcal{I}_d^c$ secured on $\{\nu \geq x_1^{(d-1)}\}$, the subtractive contribution in the Mehta expansion may then be discarded. The two remaining pieces (the linear Hermite tail $\int_u^\infty H_{d-1}(\rho x)e^{-x^2/2}\,\mathrm{d}x$ and the squared Hermite tail $\sum_j c_j^2 \int_u^\infty H_j^2(\rho x)e^{-(1+\rho^2)x^2/2}\,\mathrm{d}x$) decay at different Gaussian rates, $e^{-u^2/2}$ and $e^{-(1+\rho^2)u^2/2}$ respectively. This two-scale structure is preserved in the proposition below; the SMF bound of the next subsection collapses it to a single dominant scale. Analytically, we evaluate the linear tail by iterating the three-term Hermite recurrence, and the squared tail by substituting the Szegő envelope $H_j(\rho x) \leq (2\rho x)^j$ into each squared term and invoking Lemma 10.

**Proposition 2** (Explicit IMF upper bound on $Q_d$)**.** *For every $u > 0$ such that $\rho u \geq \sqrt{2d-1}$,*

$$\int_u^\infty Q_d(\rho x)\,e^{-x^2/2}\,\mathrm{d}x \;\leq\; \alpha_d\,\Phi_d(\rho,u)\,e^{-u^2/2} \;+\; \Psi_d(\rho,u)\,e^{-(1+\rho^2)u^2/2}\,, \tag{2.9a}$$

*where $\alpha_d$ is the dominant Mehta coefficient of Lemma 5, and the explicit polynomial–rational functions are given, with $\Lambda := 2\rho^2 - 1$, by*

$$\Phi_d(\rho, u) \; := \; \sum_{k=0}^{\lfloor (d-2)/2 \rfloor} 2\rho\,(2\Lambda)^k \,\frac{(d-2)!!}{(d-2k-2)!!}\,(2\rho u)^{d-2k-2} \; + \; \frac{\mathbb{1}_{\{d \text{ odd}\}}}{u}\,(2\Lambda)^{\frac{d-1}{2}}\,(d-2)!!\,, \tag{2.9b}$$

$$\Psi_d(\rho, u) \; := \; \frac{c_0^2}{(1+\rho^2)\,u} \; + \; \sum_{j=1}^{d-1} c_j^2 \,\frac{(2\rho)^{2j}\,u^{2j-1}}{1+\rho^2-(2j-1)/u^2}\,. \tag{2.9c}$$

*Proof of Proposition 2.* • By Szegő's bound, the largest root of $H_{d-1}$ strictly satisfies $x_1^{(d-1)} < \sqrt{2d-1}$. The proposition assumes $\rho u \geq \sqrt{2d-1}$, ensuring that $\rho x > x_1^{(d-1)}$ for all integration variables $x \geq u$. By Lemma 3, this guarantees $\mathcal{I}_d^c(\rho x) > 0$ over the entire integration domain. Returning to the exact Mehta expansion (2.3), the contribution of $-2\mathcal{I}_d^c(\rho x) H_{d-1}(\rho x)$ is therefore strictly negative. Discarding it yields a strict upper bound:

$$\int_u^\infty Q_d(\rho x) e^{-x^2/2}\,\mathrm{d}x \;\leq\; \alpha_d\,\mathcal{L}_d(u) \;+\; \mathcal{S}_d(u)\,, \tag{2.10}$$

where $\alpha_d$ is the dominant Mehta coefficient given by (2.13b), and the components are defined as:

$$\mathcal{L}_d(u) := \int_u^\infty H_{d-1}(\rho x) e^{-x^2/2}\,\mathrm{d}x\,,$$

$$\mathcal{S}_d(u) := \sum_{j=0}^{d-1} c_j^2 \int_u^\infty H_j^2(\rho x) e^{-(1+\rho^2)x^2/2}\,\mathrm{d}x\,.$$

• We derive a generalized recurrence for $J_m := \int_u^\infty H_m(\rho x) e^{-x^2/2}\,\mathrm{d}x$. Applying the identity $H_m(\rho x) = 2\rho x H_{m-1}(\rho x) - 2(m-1)H_{m-2}(\rho x)$ and evaluating the first term via integration by parts, we obtain:

$$J_m = 2\rho\, H_{m-1}(\rho u)\, e^{-u^2/2} + 2\Lambda(m-1)\, J_{m-2}\,, \qquad \text{where } \Lambda := 2\rho^2 - 1\,,$$

where we used $\frac{\mathrm{d}}{\mathrm{d}x}[H_{m-1}(\rho x)e^{-x^2/2}] = 2\rho(m-1)H_{m-2}(\rho x)e^{-x^2/2} - xH_{m-1}(\rho x)e^{-x^2/2}$. Unrolling this recurrence for $m = d-1$ produces a finite series of boundary evaluations. We bound each boundary evaluation using the Szegő envelope $H_{d-2k-2}(\rho u) \leq (2\rho u)^{d-2k-2}$. If $d-1$ is even (i.e., $d$ is odd), the recurrence bottoms out with a Gaussian tail, which we bound using Mills' ratio $\int_u^\infty e^{-x^2/2}\,\mathrm{d}x \leq u^{-1}e^{-u^2/2}$. Factoring out $e^{-u^2/2}$ yields exactly:

$$\mathcal{L}_d(u) \;\leq\; \Phi_d(\rho, u)\, e^{-u^2/2}\,,$$

where $\Phi_d$ is the polynomial-rational function defined in (2.9b).

• For the squared component, we substitute the Szegő bound $H_j(\rho x) \leq (2\rho x)^j$ directly into the integral:

$$\mathcal{S}_d(u) \;\leq\; \sum_{j=0}^{d-1} c_j^2 \int_u^\infty (2\rho x)^{2j} e^{-(1+\rho^2)x^2/2}\,\mathrm{d}x\,.$$

We evaluate each integral using Lemma 10 with parameters $a = 1+\rho^2$ and $m = 2j$. For the base case $j = 0$, we use the Mills bound $I_0(u;a) \leq (au)^{-1}e^{-au^2/2}$ of (D.1a). Factoring out the shared exponential weight $e^{-(1+\rho^2)u^2/2}$ leaves the rational function $\Psi_d(\rho, u)$ defined in (2.9c), yielding:

$$\mathcal{S}_d(u) \;\leq\; \Psi_d(\rho, u)\, e^{-(1+\rho^2)u^2/2}\,.$$

• Substituting the bounds for $\mathcal{L}_d(u)$ and $\mathcal{S}_d(u)$ back into (2.10) completes the proof:

$$\int_u^\infty Q_d(\rho x) e^{-x^2/2}\,\mathrm{d}x \;\leq\; \alpha_d\, \Phi_d(\rho, u)\, e^{-u^2/2} \;+\; \Psi_d(\rho, u)\, e^{-(1+\rho^2)u^2/2}\,.$$

□

**Theorem 3** (IMF tail bound)**.** *Let $u_{\mathrm{IMF}} = \sqrt{2d-1}/\rho$. For all $u \geq u_{\mathrm{IMF}}$,*

$$\mathbb{P}\Big\{ \sup_{\boldsymbol{\theta} \in \mathbb{S}^{d-1}} X(\boldsymbol{\theta}) > u \Big\} \;\leq\; \delta_{\mathrm{IMF}}(u) \;:=\; 2(k-1)^{\frac{d-1}{2}} \Big( \alpha_d\, \Phi_d(\rho, u)\, e^{-u^2/2} + \Psi_d(\rho, u)\, e^{-(1+\rho^2)u^2/2} \Big)\,.$$

*Proof of Theorem 3.* Apply Proposition 2 to the right-hand side of (2.4) and substitute the constant prefactor $2(k-1)^{\frac{d-1}{2}}$ from Proposition 1. □

**A sharper bound $\delta^\star_{\mathrm{IMF}}$.** The IMF bound of Theorem 3 performs two relaxations beyond $\delta_{\mathrm{exact}}$: the positivity-driven discarding of $-2\mathcal{I}^c_d\, H_{d-1}$ (Lemma 3) and the Szegő envelope $H_j(\rho x) \le (2\rho x)^j$ on the squared-Hermite tail $\sum_j c_j^2 \int_u^\infty H_j(\rho x)^2\, e^{-(1+\rho^2)x^2/2}\, \mathrm{d}x$. We introduce a sharper bound $\delta^\star_{\mathrm{IMF}}$ that performs only the first relaxation: the squared-Hermite tail is evaluated *exactly* via the closed-form recurrence of Lemma 15. The resulting $\delta^\star_{\mathrm{IMF}}$ is strictly sandwiched between $\delta_{\mathrm{exact}}$ and $\delta_{\mathrm{IMF}}$ on $[u_{\mathrm{IMF}}, \infty)$ (Theorem 4) and can replace $\delta_{\mathrm{IMF}}$ everywhere without additional analytical cost.

**Theorem 4** (Sharpened IMF tail bound). *For every $u \ge u_{\mathrm{IMF}} = \sqrt{2d-1}/\rho$,*

$$\mathbb{P}\Big\{ \sup_{\boldsymbol{\theta}\in\mathbb{S}^{d-1}} X(\boldsymbol{\theta}) > u \Big\} \;\le\; \delta^\star_{\mathrm{IMF}}(u) \;:=\; 2(k-1)^{\frac{d-1}{2}} \Big[ \alpha_d\, \Phi_d(\rho,u)\, e^{-u^2/2} \;+\; T^{\mathrm{exact}}_d(u) \Big], \tag{2.11}$$

*where the squared-Hermite tail is evaluated exactly as*

$$T^{\mathrm{exact}}_d(u) \;:=\; \frac{1}{\rho} \sum_{j=0}^{d-1} c_j^2\, K_j^\beta(\rho u), \qquad \beta \;:=\; \frac{1+\rho^2}{\rho^2} \;=\; \frac{3k-2}{k}, \tag{2.12}$$

*and each $K_j^\beta$ is given in closed form by Corollary 11. Moreover,*

$$\delta^\star_{\mathrm{IMF}}(u) \;\le\; \delta_{\mathrm{IMF}}(u) \qquad \textit{for every } u \ge u_{\mathrm{IMF}},$$

*with strict inequality at every finite $u$.*

*Proof of Theorem 4.* We follow the proof of Theorem 3, replacing only the squared-Hermite step.
• Inserting the Mehta expansion (2.3) into the right-hand side of (2.4) and discarding the strictly negative contribution $-2\mathcal{I}^c_d(\rho x)\, H_{d-1}(\rho x)$ (which is justified by Lemma 3 on $\{\rho x \ge \sqrt{2d-1}\}$) reproduces the split (2.10):

$$\int_u^\infty Q_d(\rho x)\, e^{-x^2/2}\, \mathrm{d}x \;\le\; \alpha_d\, \mathcal{L}_d(u) \;+\; \mathcal{S}_d(u),$$

with the linear-Hermite component $\mathcal{L}_d(u) = \int_u^\infty H_{d-1}(\rho x)\, e^{-x^2/2}\, \mathrm{d}x \le \Phi_d(\rho,u)\, e^{-u^2/2}$ bounded as in Proposition 2 (the recurrence-based bound on $\mathcal{L}_d$ is unchanged).
• For the squared-Hermite component $\mathcal{S}_d(u) = \sum_{j=0}^{d-1} c_j^2 \int_u^\infty H_j(\rho x)^2\, e^{-(1+\rho^2)x^2/2}\, \mathrm{d}x$, the change of variable $y = \rho x$ converts each integral to

$$\int_u^\infty H_j(\rho x)^2\, e^{-(1+\rho^2)x^2/2}\, \mathrm{d}x \;=\; \frac{1}{\rho} \int_{\rho u}^\infty H_j(y)^2\, e^{-\beta y^2/2}\, \mathrm{d}y \;=\; \frac{1}{\rho} K_j^\beta(\rho u),$$

with $\beta = (1+\rho^2)/\rho^2 = (3k-2)/k$, and Corollary 11 gives $K_j^\beta(\rho u)$ in closed form. Summing over $j$ yields $\mathcal{S}_d(u) = T^{\mathrm{exact}}_d(u)$ as defined in (2.12), with no inequality used in this step.
• Substituting the constant prefactor $2(k-1)^{(d-1)/2}$ of Proposition 1 produces (2.11). Comparing $T^{\mathrm{exact}}_d(u)$ to the Szegő-relaxed bound $\Psi_d(\rho,u)\, e^{-(1+\rho^2)u^2/2}$ of Proposition 2: by Lemma 4, $H_j(\rho x) < (2\rho x)^j$ strictly for every $j \ge 2$ and $\rho x > x_1^{(j)}$ (for $j = 0, 1$ the comparison is equality, $H_0 = 1 = (2\rho x)^0$ and $H_1(\rho x) = 2\rho x = (2\rho x)^1$). On $[u_{\mathrm{IMF}}, \infty)$, $\rho u \ge \sqrt{2d-1} > x_1^{(j)}$ for every $j \le d-1$, so every term satisfies

$$\int_u^\infty H_j(\rho x)^2\, e^{-(1+\rho^2)x^2/2}\, \mathrm{d}x \;\le\; \int_u^\infty (2\rho x)^{2j}\, e^{-(1+\rho^2)x^2/2}\, \mathrm{d}x,$$

with strict inequality whenever $j \ge 2$. Under the standing assumption $d \ge 3$, the sum (2.12) contains the index $j = 2$, contributing strictly; the remaining $j \in \{0,1\}$ contribute equality but with non-negative integrands. Therefore $T^{\mathrm{exact}}_d(u) < \Psi_d(\rho,u)\, e^{-(1+\rho^2)u^2/2}$ on $[u_{\mathrm{IMF}}, \infty)$, hence $\delta^\star_{\mathrm{IMF}}(u) < \delta_{\mathrm{IMF}}(u)$. ☐

For $k \ge 3$, $\beta = (3k-2)/k \ge 7/3 > 2$, so $\theta = (2-\beta)/\beta < 0$ in Lemma 15, and the iterated form (D.9d) carries alternating signs. The identity (2.12) remains exact pointwise, but no individual $J^\beta_{2j-2p}$ should be replaced by its absolute value during subsequent estimates; this differentiates the exact evaluation from the Szegő-envelope relaxation.

### 2.4 Simplified Mehta–Fyodorov bound (SMF)

The SMF bound applies a second relaxation beyond IMF: the Szegő envelopes $H_{d-1}(\nu) < (2\nu)^{d-1}$ and $(\nu^2+1)^{d-1}$ collapse $\Phi_d$ and the squared Hermite sum to a single monomial main term $u^{d-2}e^{-u^2/2}$ plus a faster $e^{-3u^2/4}$ remainder, at the cost of a uniform factor 2 in the leading constant (Theorem 7). Beyond the level $u_d^\star$ the remainder is dominated and the bound collapses to a single monomial (Corollary 3); this closed form is used in the inversion of Remark 1.

**Lemma 4** (Hermite envelope)**.** *For every $m \geq 2$ and every $x > x_1^{(m)}$, $0 < H_m(x) < (2x)^m$.*

*Proof of Lemma 4.* Write $H_m(x) = 2^m \prod_{j=1}^m (x - x_j^{(m)})$. For $x > x_1^{(m)}$, every factor is strictly positive, so $H_m(x) > 0$. For the upper bound, the inequality $H_m(x) < (2x)^m$ is equivalent to $\prod_{j=1}^m (1 - x_j^{(m)}/x) < 1$. Taking logarithms,

$$\sum_{j=1}^m \log(1 - x_j^{(m)}/x) \;<\; -\sum_{j=1}^m x_j^{(m)}/x \;=\; 0\,,$$

where the strict inequality uses $\log(1-t) < -t$ for $t \neq 0$ and the equality uses the vanishing sum of Hermite roots, which follows from the absence of the $x^{m-1}$ term in $H_m(x)$. □

The Hermite envelope of Lemma 4 is the only tool needed for the SMF relaxation: past the largest root, every Hermite polynomial is majorized by the monomial $(2\nu)^m$, with a strict inequality and no logarithmic corrections. We use it twice, once to collapse the linear Hermite term $\alpha_d H_{d-1}(\nu)$ to a monomial and once inside the squared Hermite sum $\sum_j c_j^2 H_j^2(\nu)$, yielding a single-monomial dominant contribution to $Q_d$ together with a controlled remainder. The next lemma states this.

**Lemma 5** (Non-asymptotic decomposition of $Q_d$)**.** *There exist explicit constants $\alpha_d, \beta_d > 0$ such that*

$$Q_d(\nu) \;=\; \alpha_d\, H_{d-1}(\nu) \;+\; \mathcal{R}_d(\nu)\,, \tag{2.13a}$$

*with the dominant coefficient given by*

$$\alpha_d \;=\; \begin{cases} \frac{1}{2}\sqrt{d/2}\, c_{d-1} c_d\, \mu_d & \text{if } d \text{ even,} \\ 1/\mu_{d-1} & \text{if } d \text{ odd,} \end{cases} \tag{2.13b}$$

*and the remainder bound, for every $\nu \geq 0$,*

$$|\mathcal{R}_d(\nu)| \;\leq\; \beta_d\, (\nu^2+1)^{d-1}\, e^{-\nu^2/2}\,. \tag{2.13c}$$

*Proof.* Deferred to Appendix A. □

The decomposition (2.13a) translates into a non-asymptotic upper bound on the Kac–Rice integrand:

$$\mathbb{E}\big[|\det(G_{d-1} - \nu I_{d-1})|\big] \;\leq\; \alpha_d\, \frac{2^{\frac{3}{2}}\Gamma((d+2)/2)}{d}\,(2\nu)^{d-1} \;+\; \frac{2^{\frac{3}{2}}\Gamma((d+2)/2)}{d}\,\beta_d\,(\nu^2+1)^{d-1}\, e^{-\nu^2/2}\,.$$

Inserting this inequality into the Kac–Rice integral (2.1b) and applying the Gaussian-type tail integrals of Lemma 10 (Appendix D.1) yields the SMF bound.

**Theorem 5** (SMF tail bound)**.** *Let $u_{\mathrm{SMF}} = 2\sqrt{d}$. For all $u \geq u_{\mathrm{SMF}}$,*

$$\mathbb{P}\Big\{\sup_{\boldsymbol{\theta}\in\mathbb{S}^{d-1}} X(\boldsymbol{\theta}) > u\Big\} \;\leq\; \delta_{\mathrm{SMF}}(u) \;:=\; 4\,\alpha_d\,(2k)^{\frac{d-1}{2}}\, u^{d-2}\, e^{-u^2/2} \;+\; 2^d\,\beta_d\,(k-1)^{\frac{d-1}{2}}\, u^{2d-3}\, e^{-3u^2/4}\,. \tag{2.14}$$

*Proof of Theorem 5.* • By Proposition 1, the excursion probability is bounded by the integral of the Mehta function:

$$\mathbb{P}\Big\{\sup_{\boldsymbol{\theta}} X(\boldsymbol{\theta}) > u\Big\} \;\leq\; 2(k-1)^{\frac{d-1}{2}} \int_u^\infty Q_d(\rho x)\, e^{-x^2/2}\, \mathrm{d}x\,.$$

Using the non-asymptotic decomposition from Lemma 5, we write $Q_d = \alpha_d H_{d-1} + \mathcal{R}_d$. This splits the corresponding integral into two parts: $I_{\text{main}} + I_{\text{rem}}$.

• The main term integral evaluates the dominant Hermite polynomial. By hypothesis, $u \geq u_{\text{SMF}} = 2\sqrt{d}$ and $\rho^2 \geq 1/2$ (since $k \geq 3$). This ensures that $\rho u \geq \sqrt{2d} > \sqrt{2d-1}$, meaning $\rho x > x_1^{(d-1)}$ for all integration variables $x \geq u$. Applying the polynomial envelope from Lemma 4, we have $0 < H_{d-1}(\rho x) < (2\rho x)^{d-1}$. Substituting this strictly positive bound into the main integral gives:

$$I_{\text{main}} \;<\; \alpha_d (2\rho)^{d-1} \int_u^\infty x^{d-1} e^{-x^2/2}\, dx\,.$$

We evaluate this using Lemma 10 with $a = 1$ and $m = d-1$. The required tail hypothesis $(d-2)/u^2 \leq 1/2$ holds safely for $u \geq 2\sqrt{d}$, yielding:

$$I_{\text{main}} \;\leq\; 2\alpha_d (2\rho)^{d-1} u^{d-2} e^{-u^2/2}\,.$$

• Using the remainder envelope (2.13c), the absolute value of the remainder integral is bounded by:

$$|I_{\text{rem}}| \;\leq\; \beta_d \int_u^\infty (\rho^2 x^2 + 1)^{d-1} e^{-(1+\rho^2)x^2/2}\, dx\,.$$

For the polynomial factor, we factor $\rho^2 x^2$ out exactly:

$$(\rho^2 x^2 + 1)^{d-1} \;=\; (\rho^2 x^2)^{d-1} \left(1 + 1/(\rho^2 x^2)\right)^{d-1} \;=\; \rho^{2(d-1)}\, x^{2(d-1)} \left(1 + 1/(\rho^2 x^2)\right)^{d-1}.$$

Since $\rho^2 \geq 1/2$ (because $k \geq 3$) and $x \geq u \geq 2\sqrt{d}$, we have $\rho^2 x^2 \geq 2d \geq 1$, so the correction factor satisfies $(1 + 1/(\rho^2 x^2))^{d-1} \leq 2^{d-1}$. Combined with $\rho \leq 1$, which gives $\rho^{2(d-1)} \leq 1$, this yields $(\rho^2 x^2 + 1)^{d-1} \leq 2^{d-1} x^{2(d-1)}$. For the exponential factor, since $k \geq 3$, we know $1 + \rho^2 \geq 3/2$. Substituting these simplifications gives:

$$|I_{\text{rem}}| \;\leq\; 2^{d-1}\beta_d \int_u^\infty x^{2(d-1)} e^{-3x^2/4}\, dx\,.$$

Applying Lemma 10 with $a = 3/2$ and $m = 2(d-1)$ (where the condition $(2d-3)/(3u^2/2) < 1$ holds for $u \geq 2\sqrt{d}$), we obtain:

$$|I_{\text{rem}}| \;\leq\; 2^{d-1}\beta_d\, u^{2d-3} e^{-3u^2/4}\,.$$

• We combine the bounded integrals and multiply by the global prefactor $2(k-1)^{\frac{d-1}{2}}$. For the main term, the identity $(2\rho)^{d-1}(k-1)^{\frac{d-1}{2}} = (2k)^{\frac{d-1}{2}}$ consolidates the constants. Summing the two appropriately scaled components yields the final bound:

$$\mathbb{P}\Big\{ \sup_{\boldsymbol{\theta}} X(\boldsymbol{\theta}) > u \Big\} \;\leq\; 4\alpha_d (2k)^{\frac{d-1}{2}} u^{d-2} e^{-u^2/2} \;+\; 2^d \beta_d (k-1)^{\frac{d-1}{2}} u^{2d-3} e^{-3u^2/4}\,. \qquad \square$$

The two-term structure of $\delta_{\text{SMF}}(u)$ reflects the two exponential scales inherited from the Mehta expansion: the $e^{-u^2/2}$ scale of the linear Hermite tail, which matches the optimal Kac–Rice decay rate, and the $e^{-3u^2/4}$ scale of the squared Hermite tail relaxed through the $(\nu^2+1)^{d-1}$ envelope. Since the latter is strictly faster, the remainder is asymptotically negligible and the bound collapses to its main term for sufficiently large $u$. The following corollary pinpoints the explicit threshold $u_d^\star \geq u_{\text{SMF}}$ beyond which this collapse occurs, at the mild price of a factor of 2 in the leading prefactor.

**Corollary 3** (Asymptotic single-term form)**.** *There exists an explicit threshold $u_d^\star \geq u_{\text{SMF}}$, depending only on $(k, d)$, such that for every $u \geq u_d^\star$ the remainder term in (2.14) is dominated by the main term, in which case*

$$\mathbb{P}\Big\{ \sup_{\boldsymbol{\theta} \in \mathbb{S}^{d-1}} X(\boldsymbol{\theta}) > u \Big\} \;\leq\; 8\,\alpha_d\,(2k)^{\frac{d-1}{2}}\, u^{d-2}\, e^{-u^2/2}\,.$$

*Explicitly, $u_d^\star$ is the smallest $u \geq 2\sqrt{d}$ satisfying $2^{d-2}\beta_d\, u^{d-1}\, e^{-u^2/4} \leq \alpha_d (2\rho)^{d-1}$.*

*Proof of Corollary 3.* • Let $R(u)$ denote the ratio of the remainder term to the main term in the SMF bound:

$$R(u) \;=\; \frac{2^d \beta_d (k-1)^{\frac{d-1}{2}}\, u^{2d-3} e^{-3u^2/4}}{4\alpha_d (2k)^{\frac{d-1}{2}}\, u^{d-2} e^{-u^2/2}}\,.$$

By grouping the constants and using the scaling parameter $\rho = \sqrt{k/(2(k-1))}$, which provides the algebraic relation $(k-1)^{\frac{d-1}{2}}/(2k)^{\frac{d-1}{2}} = (2\rho)^{-(d-1)}$, the ratio simplifies to:

$$R(u) \;=\; \frac{2^{d-2}\beta_d}{\alpha_d (2\rho)^{d-1}}\, u^{d-1} e^{-u^2/4}\,.$$

• Because the exponential decay $e^{-u^2/4}$ dominates the polynomial growth $u^{d-1}$ for large $u$, the ratio $R(u)$ decreases monotonically toward 0 as $u \to \infty$ (specifically for $u \ge \sqrt{2(d-1)}$). We require the remainder to be less than or equal to the main term, which corresponds to the condition $R(u) \le 1$.

We explicitly define the threshold $u_d^\star$ as the smallest level $u \ge u_{\mathrm{SMF}} = 2\sqrt{d}$ that satisfies this bounding condition:

$$u_d^\star \;:=\; \inf \left\{ u \ge 2\sqrt{d} \;\middle|\; u^{d-1} e^{-u^2/4} \le \frac{\alpha_d (2\rho)^{d-1}}{2^{d-2}\beta_d} \right\}.$$

Because this threshold is defined by a transcendental equation involving polynomial-exponential terms, $u_d^\star$ is the unique solution to the corresponding equality on the decaying tail, and can be evaluated directly for any fixed pair $(k, d)$.

• By definition of $u_d^\star$, for any $u \ge u_d^\star$ we have $R(u) \le 1$, so the remainder term is dominated by the main term, and the two-term bound of Theorem 5 is at most twice the main term:

$$\mathbb{P}\Big\{ \sup_{\boldsymbol{\theta}} X(\boldsymbol{\theta}) > u \Big\} \;\le\; \mathrm{Main}(u) + \mathrm{Remainder}(u) \;\le\; 2 \times \mathrm{Main}(u)\,.$$

Substituting the explicit main term yields the single-term envelope:

$$\mathbb{P}\Big\{ \sup_{\boldsymbol{\theta}} X(\boldsymbol{\theta}) > u \Big\} \;\le\; 8\alpha_d (2k)^{\frac{d-1}{2}} u^{d-2} e^{-u^2/2}\,.$$ □

## 2.5 Spectral Method (SM)

The Spectral Method does not use the Mehta–Fyodorov algebra: it combines the Ben Arous–Dembo–Guionnet large-deviation bound for the GOE spectral radius (Lemma 13) with a calibrated layer-cake split. It is an independent cross-check, incurring the same factor-2 penalty as the SMF bound and a larger validity threshold (Theorem 7).

Write $M_{d-1} := \max_i |\mu_i|$ for the spectral radius of $G_{d-1}$. A two-scale layer-cake split at $R = \sqrt{|\nu|\sqrt{d-1}}$ (well below $|\nu|$, well above the edge $\sim 2\sqrt{d-1}$) replaces the crude $|\mu_i - \nu| \le 2|\nu|$ by $|\mu_i - \nu| \le (1 + R/|\nu|)|\nu|$ on the bulk event $\{M_{d-1} \le R\}$, the tail $\{M_{d-1} > R\}$ being controlled by the Ben Arous–Dembo–Guionnet bound (Lemma 13); this removes the $2^{d-1}$ penalty of the naive argument.

**Proposition 4** (SM bound on the expected absolute characteristic polynomial)**.** *Let $G_{d-1} \sim \mathrm{GOE}(d-1)$ with eigenvalues $\mu_1, \ldots, \mu_{d-1}$. For every threshold $R = R(d, \nu)$ satisfying $4\sqrt{2(d-1)} \le R \le |\nu|$,*

$$\mathbb{E}[|\det(G_{d-1} - \nu\, I_{d-1})|] \;=\; \mathbb{E}\Big[\prod_{i=1}^{d-1} |\mu_i - \nu|\Big] \;\le\; |\nu|^{d-1} \Big[ \big(1 + \tfrac{R}{|\nu|}\big)^{d-1} + (d+1)\, 2^{d-1}\, e^{-2R^2/9} \Big]. \quad (2.15a)$$

*In particular, for $|\nu| \ge 32\sqrt{d-1}$ and the canonical choice $R = \sqrt{|\nu|\sqrt{d-1}}$,*

$$\mathbb{E}\Big[\prod_{i=1}^{d-1} |\mu_i - \nu|\Big] \;\le\; |\nu|^{d-1}\big(1 + \eta_d(\nu)\big), \quad \eta_d(\nu) := \Big(1 + \big(\tfrac{\sqrt{d-1}}{|\nu|}\big)^{\frac{1}{2}}\Big)^{d-1} - 1 + (d+1)\, 2^{d-1}\, e^{-\frac{2|\nu|\sqrt{d-1}}{9}}, \quad (2.15b)$$

*with $\eta_d(\nu) \to 0$ as $|\nu|/\sqrt{d-1} \to \infty$.*

*Proof of Proposition 4.* Fix any admissible threshold $R \in [4\sqrt{2(d-1)}, |\nu|]$ and decompose the expectation according to $\{M_{d-1} \le R\}$ versus $\{M_{d-1} > R\}$:

$$\mathbb{E}\Big[\prod_{i=1}^{d-1} |\mu_i - \nu|\Big] \;=\; \mathbb{E}\Big[\prod_{i=1}^{d-1} |\mu_i - \nu| \, \mathbb{1}_{\{M_{d-1} \le R\}}\Big] + \mathbb{E}\Big[\prod_{i=1}^{d-1} |\mu_i - \nu| \, \mathbb{1}_{\{M_{d-1} > R\}}\Big] .$$

• On $\{M_{d-1} \le R\}$, $|\mu_i - \nu| \le |\mu_i| + |\nu| \le R + |\nu|$, hence the bulk contribution is at most $(R + |\nu|)^{d-1} = |\nu|^{d-1}(1 + R/|\nu|)^{d-1}$.

• On $\{M_{d-1} > R\}$, the triangle inequality yields $|\mu_i - \nu| \le M_{d-1} + |\nu|$, and we distinguish two subcases:

$$\big(|\nu| + M_{d-1}\big)^{d-1} \mathbb{1}_{\{M_{d-1} > R\}} \;\le\; (2|\nu|)^{d-1} \mathbb{1}_{\{R < M_{d-1} \le |\nu|\}} + (2M_{d-1})^{d-1} \mathbb{1}_{\{M_{d-1} > |\nu|\}} .$$

By Lemma 13 (whose hypothesis holds since $R \ge 4\sqrt{2(d-1)}$), the first piece contributes at most by a term $(2|\nu|)^{d-1} e^{-2R^2/9}$. For the second piece, the layer-cake formula applied to $Z = (2M_{d-1})^{d-1} \mathbb{1}_{\{M_{d-1} > |\nu|\}}$ gives

$$\mathbb{E}[Z] \;=\; 2^{d-1} |\nu|^{d-1} \mathbb{P}\{M_{d-1} > |\nu|\} + (d-1)\, 2^{d-1} \int_{|\nu|}^{\infty} s^{d-2}\, \mathbb{P}\{M_{d-1} > s\} \, \mathrm{d}s ,$$

and, since $|\nu| \ge R \ge 4\sqrt{2(d-1)}$, Lemma 13 again applies on $[|\nu|, \infty)$.

By Lemma 10 with $a = 4/9$, $m = d-2$, the integral is at most $|\nu|^{d-1} e^{-2|\nu|^2/9}$ (check that its hypothesis $(d-3)/(4|\nu|^2/9) < 1/14$ holds since $|\nu|^2 \ge 32(d-1)$), yielding

$$\mathbb{E}[Z] \le d\, 2^{d-1}\, |\nu|^{d-1}\, e^{-2|\nu|^2/9} .$$

Combining the two subcases and bounding $e^{-2|\nu|^2/9} \le e^{-2R^2/9}$ (since $R \le |\nu|$),

$$\mathbb{E}\big[(|\nu| + M_{d-1})^{d-1} \mathbb{1}_{\{M_{d-1} > R\}}\big] \;\le\; (d+1)\, 2^{d-1}\, |\nu|^{d-1}\, e^{-2R^2/9} .$$

• Adding both previous terms yields (2.15a). The canonical choice $R = \sqrt{|\nu|\sqrt{d-1}}$ is admissible precisely when $|\nu| \ge 32\sqrt{d-1}$ (equivalently $R \ge 4\sqrt{2(d-1)}$), in which case $R/|\nu| = (\sqrt{d-1}/|\nu|)^{1/2}$ and $2R^2/9 = 2|\nu|\sqrt{d-1}/9$, giving (2.15b). □

Proposition 4 gives a pointwise bound on the expected absolute determinant inside the Kac–Rice integral, with the remaining $x$-dependence concentrated in the leading monomial $(\rho x)^{d-1}$. Integrating against the standard Gaussian weight $\varphi(x)$ then reduces to the Gaussian moment integrals of Lemma 10. The resulting tail bound inherits the polynomial–exponential envelope $u^{d-2} e^{-u^2/2}$, which already has the optimal asymptotic decay rate; the multiplicative factor $1 + \eta_d(\rho, u)$ measures the cost of the layer-cake split and vanishes whenever $\rho u/\sqrt{d-1} \to \infty$.

**Theorem 6** (SM tail bound)**.** *Let $u_{\mathrm{SM}} = 32\frac{\sqrt{d-1}}{\rho} = 32\sqrt{2(d-1)\frac{k-1}{k}}$. For all $u \ge u_{\mathrm{SM}}$,*

$$\mathbb{P}\Big\{\sup_{\boldsymbol{\theta} \in \mathbb{S}^{d-1}} X(\boldsymbol{\theta}) > u\Big\} \;\le\; \delta_{\mathrm{SM}}(u) \;:=\; 2\frac{\sqrt{2}}{\Gamma(d/2)} \Big(\frac{k}{2}\Big)^{\frac{d-1}{2}} u^{d-2}\, e^{-u^2/2} \big(1 + \eta_d(\rho, u)\big) , \tag{2.17}$$

*where $\eta_d(\rho, u)$ is given by (1.6a) and is such that $\eta_d(\rho, u) \to 0$ as $\rho u/\sqrt{d-1} \to \infty$.*

*Proof of Theorem 6.* • We apply Proposition 4 pointwise to the Kac–Rice reduction (2.1b), setting $\nu = \rho x$ and $R(x) = \sqrt{\rho x \sqrt{d-1}}$. For any level $u \ge u_{\mathrm{SM}} = 32\sqrt{d-1}/\rho$ and every integration variable $x \ge u$, we have $\rho x \ge 32\sqrt{d-1}$. This guarantees that the admissibility condition $R(x) \ge 4\sqrt{2(d-1)}$ is satisfied. Therefore, the expected absolute determinant is bounded by:

$$\mathbb{E}[|\det(G_{d-1} - \rho x\, I_{d-1})|] \;\le\; (\rho x)^{d-1}\big(1 + \eta_d(\rho x)\big) ,$$

where $\eta_d(\rho x)$ is defined in (2.15b).

• Since $\eta_d(\cdot)$ is non-increasing, $\eta_d(\rho x) \le \eta_d(\rho u)$ for all $x \ge u$; write $\eta(u) := \eta_d(\rho u)$. Substituting into the Kac–Rice integral yields:

$$\mathbb{P}\Big\{\sup_{\boldsymbol{\theta}} X(\boldsymbol{\theta}) > u\Big\} \;\le\; C_{k,d}\,\rho^{d-1}\big(1+\eta(u)\big)\int_u^\infty x^{d-1}\varphi(x)\,\mathrm{d}x\,. \tag{2.18a}$$

• To evaluate the remaining integral, we apply Lemma 10 with $a = 1$ and $m = d-1$. The required hypothesis $(d-2)/u^2 \le 1/2$ holds safely since $u \ge u_{\mathrm{SM}} \ge 2\sqrt{d-1}$. This gives:

$$\int_u^\infty x^{d-1}\varphi(x)\,\mathrm{d}x \;\le\; \frac{2}{\sqrt{2\pi}}u^{d-2}e^{-u^2/2}\,.$$

Finally, we substitute the geometric volume prefactor $C_{k,d} = 2\sqrt{\pi}(k-1)^{\frac{d-1}{2}}/\Gamma(d/2)$ into our bound. Using the identity $\rho^{d-1}(k-1)^{\frac{d-1}{2}} = (k/2)^{\frac{d-1}{2}}$, the constants consolidate:

$$\mathbb{P}\Big\{\sup_{\boldsymbol{\theta}} X(\boldsymbol{\theta}) > u\Big\} \;\le\; \frac{2\sqrt{2}\,(k/2)^{\frac{d-1}{2}}}{\Gamma(d/2)}\,u^{d-2}\,e^{-u^2/2}\,\big(1+\eta(u)\big)\,. \qquad \square$$

## 2.6 Pointwise domination of the IMF bound

**Theorem 7** (Domination of the SMF bound by the IMF bound)**.** *For every $k \ge 3$ and $d \ge 3$,*

$$\delta_{\mathrm{IMF}}(u) \;\le\; \delta_{\mathrm{SMF}}(u) \qquad \textit{for every } u \ge u_{\mathrm{SMF}} = 2\sqrt{d}\,. \tag{2.19}$$

The proof uses the geometric envelope of Lemma 7; the prefactor identity of Lemma 6, also recorded here, enters the secondary SM comparison of Remark 2.

**Lemma 6** (Asymptotic-prefactor identity $P_{\mathrm{SMF}} = P_{\mathrm{SM}}$)**.** *For every integer $d \ge 1$ and every $k \ge 2$,*

$$4\,\alpha_d\,(2k)^{\frac{d-1}{2}} \;=\; 2\,\frac{\sqrt{2}}{\Gamma(d/2)}\left(\frac{k}{2}\right)^{\frac{d-1}{2}}, \tag{2.20}$$

*or equivalently $\alpha_d = 2^{-(d-1/2)}/\Gamma(d/2)$.*

*Proof.* We verify $\alpha_d \cdot 2^{d-1/2}\Gamma(d/2) = 1$ for both parities of $d$.

For $d$ odd, write $d = 2m+1$ ($m \ge 0$). Then by Lemma 5 and the closed form $\mu_{2m} = \sqrt{2\pi}(2m)!/m!$ recalled in (1.6),

$$\alpha_d \;=\; \frac{1}{\mu_{2m}} \;=\; \frac{m!}{\sqrt{2\pi}\,(2m)!}\,, \qquad \Gamma(d/2) \;=\; \Gamma\big(m+\tfrac{1}{2}\big) \;=\; \frac{(2m-1)!!\,\sqrt{\pi}}{2^m}\,.$$

Substituting and using the factorisation $(2m)! = 2^m\,m!\,(2m-1)!!$,

$$\alpha_d \cdot 2^{d-1/2}\Gamma(d/2) \;=\; \frac{m!}{\sqrt{2\pi}\,(2m)!}\cdot 2^{2m+1/2}\cdot\frac{(2m-1)!!\sqrt{\pi}}{2^m} \;=\; \frac{m!\,2^m\,(2m-1)!!}{(2m)!} \;=\; 1\,.$$

For $d$ even, write $d = 2m$ ($m \ge 1$). Then $\alpha_d = \frac{1}{2}\sqrt{m}\,c_{2m-1}c_{2m}\,\mu_{2m}$ with $c_j = (2^j j!\sqrt{\pi})^{-1/2}$ and $\Gamma(d/2) = (m-1)!$. A direct computation yields $c_{2m-1}c_{2m} = 1/[\sqrt{\pi}\,2^{2m-1/2}\sqrt{(2m-1)!\,(2m)!}]$ and combining with $\mu_{2m} = \sqrt{2\pi}\,(2m)!/m!$ produces

$$\alpha_d \;=\; \frac{\sqrt{m}\,\sqrt{2\pi}\,(2m)!}{2\sqrt{\pi}\,2^{2m-1/2}\sqrt{(2m-1)!(2m)!}\,m!} \;=\; \frac{\sqrt{m\,(2m)!\,/(2m-1)!}}{2^{2m}\,m!} \;=\; \frac{\sqrt{2m^2}}{2^{2m}\,m!} \;=\; \frac{1}{2^{2m-1/2}\,(m-1)!}\,,$$

using $(2m)!/(2m-1)! = 2m$. Hence $\alpha_d \cdot 2^{d-1/2}\Gamma(d/2) = 2^{-(2m-1/2)}\,2^{2m-1/2}\,(m-1)!/(m-1)! = 1$. $\square$

**Lemma 7** (Geometric envelope on $\Phi_d$)**.** *For every $k \ge 3$, $d \ge 3$, and every $u \ge u_{\mathrm{SMF}} = 2\sqrt{d}$,*

$$\Phi_d(\rho, u) \;\le\; 2\,(2\rho)^{d-1}\,u^{d-2}\,. \tag{2.21}$$

*Proof.* Since $\Lambda = 2\rho^2 - 1 = 1/(k-1)$ (recalled below (1.6)), we have $2\Lambda = 2/(k-1)$, and we set

$$q \;:=\; \frac{2\Lambda}{(2\rho u)^2} \;=\; \frac{2}{(k-1)\,(2\rho u)^2}\,.$$

On $\{u \geq 2\sqrt{d}\}$, $\rho^2 \geq 1/2$ for $k \geq 3$, hence $(2\rho u)^2 \geq 4\rho^2 \cdot 4d \geq 8d$, and therefore

$$(d-2)\,q \;\leq\; \frac{2(d-2)}{8d\,(k-1)} \;=\; \frac{d-2}{4d\,(k-1)} \;\leq\; \frac{1}{4(k-1)} \;\leq\; \frac{1}{8}\,.$$

The leading term of $\Phi_d$ (the $\ell = 0$ summand of (2.9b)) equals $2\rho\,(2\rho u)^{d-2} = (2\rho)^{d-1}u^{d-2}$. The $\ell$-th summand for $\ell \geq 1$, divided by the leading term, equals $(d-2)(d-4)\cdots(d-2\ell)\,q^{\ell}$, a product of $\ell$ factors each at most $(d-2)q \leq 1/8$; hence

$$\sum_{\ell\geq 1} \frac{\ell\text{-th summand}}{\text{leading}} \;\leq\; \sum_{\ell\geq 1}(1/8)^{\ell} \;=\; \frac{1}{7}\,.$$

Similarly, for $d$ odd, the trailing term $\mathbb{1}_{\{d\text{ odd}\}}\,u^{-1}\,(2\Lambda)^{(d-1)/2}\,(d-2)!!$ divided by the leading is $q^{(d-1)/2}\,(d-2)!! = \prod_{j=1}^{(d-1)/2}(d-2j)\,q \leq (1/8)^{(d-1)/2} \leq 1/8$ (the last bound uses $(d-1)/2 \geq 1$ for $d \geq 3$). Summing,

$$\Phi_d(\rho,u)/\big((2\rho)^{d-1}u^{d-2}\big) \;\leq\; 1 + \tfrac{1}{7} + \tfrac{1}{8} \;=\; \tfrac{71}{56} \;\leq\; 2\,,$$

which is (2.21). □

*Proof of Theorem 7.* We bound the IMF main and residual terms separately on $[u_{\mathsf{SMF}}, \infty)$.

• *Main term.* On $[u_{\mathsf{SMF}}, \infty)$, Lemma 7 gives $\Phi_d(\rho,u) \leq 2(2\rho)^{d-1}u^{d-2}$. Multiply by $2(k-1)^{(d-1)/2}\alpha_d e^{-u^2/2}$ and use $\rho^{d-1}(k-1)^{(d-1)/2} = (k/2)^{(d-1)/2}$:

$$\underbrace{2(k-1)^{\frac{d-1}{2}}\,\alpha_d\,\Phi_d(\rho,u)\,e^{-u^2/2}}_{\text{IMF main}} \;\leq\; 4\,\alpha_d\,(2\rho)^{d-1}\,(k-1)^{\frac{d-1}{2}}\,u^{d-2}\,e^{-u^2/2}$$
$$=\; 4\,\alpha_d\,(2k)^{\frac{d-1}{2}}\,u^{d-2}\,e^{-u^2/2} \;=\; \underbrace{\delta^{\mathrm{main}}_{\mathsf{SMF}}(u)}_{\text{first term of (2.14)}}\,.$$

• *Residual.* On $[u_{\mathsf{SMF}}, \infty)$, $u^2 \geq 4d$ so $(2j-1)/u^2 \leq (2d-3)/(4d) \leq 1/2$ for every $j \leq d-1$; combined with $1+\rho^2 \geq 3/2$ (since $k \geq 3$), the rational factor in (2.9c) satisfies

$$\frac{1}{1+\rho^2-(2j-1)/u^2} \;\leq\; \frac{1}{1+\rho^2-1/2} \;\leq\; 1\,.$$

Hence

$$\Psi_d(\rho,u) \;\leq\; \frac{c_0^2}{(1+\rho^2)\,u} + \sum_{j=1}^{d-1} c_j^2\,(2\rho)^{2j}\,u^{2j-1} \;\leq\; u^{2d-3}\sum_{j=0}^{d-1} c_j^2\,(2\rho)^{2j}\,,$$

using $u^{2j-1} \leq u^{2d-3}$ for $j \leq d-1$ and $u \geq 1$, the trivial bound $1/[(1+\rho^2)u] \leq u^{2d-3}$, and $1/(1+\rho^2) \leq 1 \leq (2\rho)^0 = 1$ to consolidate the $j=0$ term. Comparing the per-$j$ Szegő-style coefficient $c_j^2(2\rho)^{2j} = c_j^2\,4^j\rho^{2j}$ to the entry $c_j^2\,2^j\,((2j)!/j!)^2$ of $S_d$ in (1.6): since $(2j)!/j! = (j+1)(j+2)\cdots(2j) \geq 2^j$ for $j \geq 1$, $((2j)!/j!)^2 \geq 4^j$, and using $\rho \leq 1$, $(2\rho)^{2j} = 4^j\rho^{2j} \leq 4^j \leq 2^j\cdot 4^j \leq 2^j((2j)!/j!)^2$, hence $c_j^2(2\rho)^{2j} \leq c_j^2\,2^j((2j)!/j!)^2$ termwise (for $j=0$ the comparison reads $c_0^2 \leq c_0^2$ with equality), giving $\sum_j c_j^2(2\rho)^{2j} \leq S_d \leq \beta_d$. Therefore

$$\underbrace{2(k-1)^{\frac{d-1}{2}}\,\Psi_d(\rho,u)\,e^{-(1+\rho^2)u^2/2}}_{\text{IMF residual}} \;\leq\; 2(k-1)^{\frac{d-1}{2}}\beta_d\,u^{2d-3}\,e^{-(1+\rho^2)u^2/2}\,.$$

Comparing to $\delta^{\mathrm{rem}}_{\mathsf{SMF}}(u) := 2^d\beta_d(k-1)^{(d-1)/2}u^{2d-3}e^{-3u^2/4}$, the ratio of exponentials is $e^{-(1+\rho^2)u^2/2+3u^2/4} = e^{-\Lambda u^2/4}$, which lies in $(0,1]$ for $\Lambda = 2\rho^2-1 = 1/(k-1) > 0$ (i.e., $k \geq 3$). The ratio of polynomial prefactors is $2/2^d = 2^{1-d} \leq 1$ for $d \geq 1$. Hence the IMF residual is bounded by $\delta^{\mathrm{rem}}_{\mathsf{SMF}}$ on $[u_{\mathsf{SMF}}, \infty)$.

Adding the two pieces, $\delta_{\mathsf{IMF}}(u) \leq \delta^{\mathrm{main}}_{\mathsf{SMF}}(u) + \delta^{\mathrm{rem}}_{\mathsf{SMF}}(u) = \delta_{\mathsf{SMF}}(u)$ on $[u_{\mathsf{SMF}}, \infty)$, which is (2.19). □

**Remark 2** (SM-branch domination: a secondary observation)**.** *On its own validity range* $[u_{\mathrm{SM}}, \infty)$ *the independent SM bound is also dominated by* $\delta_{\mathrm{SMF}}$ *(hence, by Theorem 7, by* $\delta_{\mathrm{IMF}}$*) for every* $(k, d)$ *of practical interest, although, unlike Theorem 7, this is* not *claimed uniformly in* $(k, d)$*. Indeed, by Lemma 6 the SM and SMF main coefficients are equal:*

$$\delta^{\mathrm{main}}_{\mathrm{SMF}}(u) \;=\; 2\,\frac{\sqrt{2}}{\Gamma(d/2)}\,(k/2)^{(d-1)/2}\,u^{d-2}\,e^{-u^2/2} \;=\; \delta_{\mathrm{SM}}(u)/(1+\eta_d(\rho,u))\,.$$

*Therefore*

$$\delta_{\mathrm{SM}}(u) - \delta_{\mathrm{SMF}}(u) \;=\; \delta^{\mathrm{main}}_{\mathrm{SMF}}(u)\,\eta_d(\rho,u) \;-\; \delta^{\mathrm{rem}}_{\mathrm{SMF}}(u)\,.$$

*It suffices to show* $\delta^{\mathrm{rem}}_{\mathrm{SMF}}(u) \le \delta^{\mathrm{main}}_{\mathrm{SMF}}(u)\,\eta_d(\rho,u)$ *on* $[u_{\mathrm{SM}}, \infty)$*. Dividing by* $\delta^{\mathrm{main}}_{\mathrm{SMF}}(u) > 0$*, this reads*

$$h(u) \;:=\; \frac{C\,u^{d-1}\,e^{-u^2/4}}{\eta_d(\rho,u)} \;\le\; 1 \qquad \text{on } [u_{\mathrm{SM}},\infty), \qquad C \;:=\; \frac{2^d\,\beta_d\,(k-1)^{(d-1)/2}}{4\,\alpha_d\,(2k)^{(d-1)/2}}\,, \tag{2.22}$$

*where the polynomial-prefactor reduction uses* $u^{2d-3}/u^{d-2} = u^{d-1}$ *and the exponential reduction uses* $e^{-3u^2/4}/e^{-u^2/2} = e^{-u^2/4}$*.*

Lower bound on $\eta_d(\rho,u)$. *Bernoulli's inequality* $(1+\varepsilon)^{d-1} \ge 1 + (d-1)\varepsilon$ *for* $\varepsilon \ge 0$ *and* $d \ge 2$*, applied with* $\varepsilon = \sqrt{\sqrt{d-1}/(\rho u)}$ *to the first term of (1.6a), yields the u-dependent lower bound*

$$\eta_d(\rho,u) \;\ge\; (d-1)\left(\sqrt{d-1}/(\rho u)\right)^{1/2} \;=\; \frac{(d-1)^{5/4}}{(\rho u)^{1/2}}\,, \qquad \text{for every } u > 0 \text{ and } d \ge 2. \tag{2.23}$$

Monotonicity of $h$. *Substituting (2.23) into (2.22),*

$$h(u) \;\le\; \frac{C\,(\rho u)^{1/2}}{(d-1)^{5/4}}\,u^{d-1}\,e^{-u^2/4} \;=\; \frac{C\,\rho^{1/2}}{(d-1)^{5/4}}\,u^{d-1/2}\,e^{-u^2/4}\,.$$

*The function* $u \mapsto u^{d-1/2}\,e^{-u^2/4}$ *has derivative* $u^{d-3/2}e^{-u^2/4}\big[(d-1/2) - u^2/2\big]$ *which is non-positive for* $u^2 \ge 2d-1$*. Since* $u^2_{\mathrm{SM}} = 1024(d-1)/\rho^2 \ge 1024(d-1) \ge 2d-1$ *for* $d \ge 2$*, the function is decreasing on* $[u_{\mathrm{SM}}, \infty)$*. Therefore*

$$\sup_{u \ge u_{\mathrm{SM}}} h(u) \;\le\; h(u_{\mathrm{SM}})_{upper} \;:=\; \frac{C\,\rho^{1/2}}{(d-1)^{5/4}}\,u_{\mathrm{SM}}^{d-1/2}\,e^{-u^2_{\mathrm{SM}}/4}\,. \tag{2.24}$$

Doubly-exponential bound at $u_{\mathrm{SM}}$. *Using* $1/\rho \le \sqrt{2}$*,* $1/\rho^2 \ge 1$*:*

$$u_{\mathrm{SM}}^{d-1/2} \;=\; \left(32\sqrt{d-1}/\rho\right)^{d-1/2} \;\le\; \left(32\sqrt{2(d-1)}\right)^{d-1/2} \;=\; (2048)^{(d-1/2)/2}(d-1)^{(d-1/2)/2}\,,$$

$$e^{-u^2_{\mathrm{SM}}/4} \;=\; e^{-256(d-1)/\rho^2} \;\le\; e^{-256(d-1)}\,.$$

*Hence*

$$u_{\mathrm{SM}}^{d-1/2}\,e^{-u^2_{\mathrm{SM}}/4} \;\le\; (2048)^{(d-1/2)/2}\,(d-1)^{(d-1/2)/2}\,e^{-256(d-1)}\,. \tag{2.25}$$

Stirling control of $\beta_d/\alpha_d$. *By Lemma 6,* $\alpha_d = 2^{-(d-1/2)}/\Gamma(d/2)$*. Stirling's upper bound at* $z = d/2$ *gives* $\Gamma(d/2) \le \sqrt{2\pi}\,(d/2)^{(d-1)/2}e^{-d/2+1/(6d)}$*. Combined with* $2^{d-1/2}\,(d/2)^{(d-1)/2} \le d^{d/2}\cdot 2^{(d-1)/2}$*, this yields* $1/\alpha_d \le d^{d/2}\cdot 2^{(d-1)/2}$*, so*

$$\alpha_d \;\ge\; d^{-d/2}\cdot 2^{-(d-1)/2}\,. \tag{2.26}$$

*For the numerator, by the definition of* $\beta_d$ *in the proof of Lemma 5,*

$$\beta_d \;\le\; S_d \;+\; \sqrt{d/2}\,c_{d-1}c_d\,\frac{(2d-2)!}{(d-1)!}\,2^{d-1}\,\tilde{B}_d\,,$$

*with* $c_j = (2^j j!\sqrt{\pi})^{-1/2}$*,* $S_d = \sum_{j=0}^{d-1} c_j^2\,2^j\,((2j)!/j!)^2$*, and* $\tilde{B}_d = \max((2d)!\,2^{d+1}/d!,\, B'_d)$*. Each ingredient is bounded by a super-exponential of* $d\log d$*: using* $(2j)!/j! \le (2j)! \le (2j)^{2j}$ *and* $c_j^2\,2^j \le 1/(j!\sqrt{\pi})$*, one* *obtains* $S_d \le e^{c\,d\log d}$ *for some absolute* $c$*; similarly* $(2d-2)!/(d-1)! \le (2d-2)^{d-1}$ *and* $\tilde{B}_d \le (2d)^{2d+1}$*, and* $\sqrt{d/2}\,c_{d-1}c_d\,2^{d-1} \le d^d$ *for* $d$ *large. Multiplying, there is an absolute constant* $A > 0$ *such that*

$$\beta_d \;\le\; e^{A\,d\log d} \qquad \text{for every } d \ge 3. \tag{2.27}$$

*Combining* (2.26) *and* (2.27)*,*

$$\beta_d/\alpha_d \;\le\; e^{A\,d\log d}\cdot d^{d/2}\cdot 2^{(d-1)/2} \;\le\; e^{A'\,d\log d} \tag{2.28}$$

*for some absolute* $A' > 0$ *and every* $d \ge 3$*.*

Closing the estimate. *The constant* $C$ *in* (2.22) *satisfies* $C \le c_1\,(2(k-1))^{(d-1)/2}\beta_d/\alpha_d \le c_1\,(2(k-1))^{(d-1)/2}\,e^{A'd\log d}$ *for an absolute* $c_1 > 0$*. Combining with* (2.25)*, the bound* $u_{\mathrm{SM}}^{d-1/2} \le (2048(d-1))^{(d-1/2)/2}$*, and* $1/\rho^2 \ge 1$*,*

$$h(u_{\mathrm{SM}})_{upper} \;\le\; \exp\Big[\underbrace{(A'+\tfrac{1}{2})\,d\log d + \tfrac{d-1}{2}\log\big(2(k-1)\big) + O(d)}_{\text{prefactor growth}} \;-\; \underbrace{256(d-1)/\rho^2}_{\ge 256(d-1)}\Big].$$

*The subtracted term is* linear *in* $d$*, whereas the prefactor growth is of order* $d\log d$*. Hence the bracket is negative (so that* $h(u_{\mathrm{SM}}) \le 1$*, and by the monotonicity* (2.24) $h(u) \le 1$ *on all of* $[u_{\mathrm{SM}},\infty)$*, i.e.* $\delta_{\mathrm{SMF}}(u) \le \delta_{\mathrm{SM}}(u)$ *there) for every* $d$ *below the threshold* $d^\star(k) := \exp\big(256/(A'+\tfrac{1}{2}) + O(1)\big)$*, which far exceeds any dimension of practical interest. Because* $d\log d$ *ultimately overtakes the linear term, we do* not *claim the inequality for literally all* $(k,d)$*: the competition between the super-exponential gain* $e^{-256(d-1)/\rho^2}$ *at* $u_{\mathrm{SM}}$ *and the* $e^{O(d\log d)}$ *prefactor growth is genuine. This non-uniformity is immaterial for the rest of the paper: the master failure probability* $\delta_{\min} = 2\,\delta_{\mathrm{IMF}}$ *of Theorem 1 is unconditional (it is the IMF bound of Theorem 3 combined with the two-sided symmetry), and for any* $(k,d)$ *outside the certified range one invokes the unconditional fallback* $\delta_{\min} \le \min(2\delta_{\mathrm{IMF}}, 2\delta_{\mathrm{SM}})$ *on* $[u_{\mathrm{SM}},\infty)$ *recorded in Corollary 5.*

**Corollary 5** (Master bound)**.** *For every* $k \ge 3$ *and* $d \ge 3$*, the master failure probability* (1.7a) *equals the IMF branch on its entire validity range,*

$$\forall u \ge u_{\mathrm{IMF}}\,, \qquad \delta_{\min}(u) \;=\; 2\,\delta_{\mathrm{IMF}}(u)\,, \tag{2.29a}$$

*and this is the pointwise smaller of the two Mehta–Fyodorov candidates:* $2\,\delta_{\mathrm{IMF}}(u) \le 2\,\delta_{\mathrm{SMF}}(u)$ *on* $[u_{\mathrm{SMF}},\infty)$ *by Theorem 7. On the SM range* $[u_{\mathrm{SM}},\infty)$ *one further has* $2\,\delta_{\mathrm{IMF}}(u) \le 2\,\delta_{\mathrm{SM}}(u)$ *for every* $(k,d)$ *of practical interest (Remark 2); in all cases the conservative bound*

$$\delta_{\min}(u) \;\le\; \min\big(2\,\delta_{\mathrm{IMF}}(u),\, 2\,\delta_{\mathrm{SM}}(u)\big) \qquad \text{for } u \ge u_{\mathrm{SM}} \tag{2.29b}$$

*remains unconditional. In particular,*

$$\forall u \ge u_{\mathrm{IMF}}\,, \qquad \mathbb{P}\big\{\Gamma_{1,1} > u\big\} \;\le\; 2\,\delta_{\mathrm{IMF}}(u)\,,$$

*which is the bound used in Theorem 1.*

*Proof.* The bound $\mathbb{P}\{\Gamma_{1,1} > u\} \le 2\,\delta_{\mathrm{IMF}}(u)$ is Theorem 3 combined with the two-sided symmetry (1.4c), hence unconditional; this is (2.29a), as $\delta_{\min}$ is defined by $\delta_{\min} = 2\,\delta_{\mathrm{IMF}}$ in (1.7a). Theorem 7 gives $\delta_{\mathrm{IMF}} \le \delta_{\mathrm{SMF}}$ on $[u_{\mathrm{SMF}},\infty)$ for every $k \ge 3$, $d \ge 3$, so $2\,\delta_{\mathrm{IMF}}$ is the smaller of the two MF candidates there; the comparison with the SM branch on $[u_{\mathrm{SM}},\infty)$ is Remark 2, and where it is not invoked the piecewise minimum (2.29b) is unconditional. □

# 3 Statistical analysis of Tensor PCA

## 3.1 Geometric domination and rank reduction

Non-asymptotic statistical control of the profile MLE uses two deterministic ingredients: a geometric (Tube Method) inequality bounding the estimation error by the noise supremum on the feasible manifold, and a rank-reduction inequality reducing this constrained noise supremum to the unconstrained rank-one Shub–Smale supremum.

**Proposition 6** (MLE formulation)**.** *Under the model* (1.1) *and for any closed subset* $\mathcal{C} \subseteq \mathfrak{S}_R$*, the Maximum Likelihood Estimator of* $(\lambda, \boldsymbol{\sigma}^\star)$ *over* $\mathbb{R} \times \mathcal{C}$ *satisfies*

$$\hat{\boldsymbol{\sigma}} \in \arg\max_{\boldsymbol{\sigma}\in\mathcal{C}} \langle \boldsymbol{Y}, \boldsymbol{\sigma} \rangle_{\mathcal{T}}\,, \qquad \hat{\lambda} = \langle \boldsymbol{Y}, \hat{\boldsymbol{\sigma}} \rangle_{\mathcal{T}}\,.$$

*The profile MLE* (1.3) *corresponds to* $\mathcal{C} = \{\boldsymbol{\sigma} \in \mathfrak{S}_R : \kappa(\boldsymbol{\sigma}) \geq \kappa\}$*.*

*Proof of Proposition 6.* The negative log-likelihood of $(\lambda, \boldsymbol{\sigma})$ given $\boldsymbol{Y}$ is, up to additive constants,

$$\mathcal{L}(\lambda, \boldsymbol{\sigma}) \;=\; \|\boldsymbol{Y} - \lambda\boldsymbol{\sigma}\|_F^2 \;=\; \|\boldsymbol{Y}\|_F^2 - 2\lambda\langle \boldsymbol{Y}, \boldsymbol{\sigma}\rangle_{\mathcal{T}} + \lambda^2\,,$$

where we used $\|\boldsymbol{\sigma}\|_F^2 = 1$. Minimizing in $\lambda$ for fixed $\boldsymbol{\sigma}$ gives $\hat{\lambda}(\boldsymbol{\sigma}) = \langle \boldsymbol{Y}, \boldsymbol{\sigma}\rangle_{\mathcal{T}}$. Substituting back yields the profile cost $\|\boldsymbol{Y}\|_F^2 - \langle \boldsymbol{Y}, \boldsymbol{\sigma}\rangle_{\mathcal{T}}^2$, whose minimization over $\mathcal{C}$ is equivalent to maximizing $|\langle \boldsymbol{Y}, \boldsymbol{\sigma}\rangle_{\mathcal{T}}|$ over $\mathcal{C}$. Since $\mathfrak{S}_R$ is closed under sign change $\boldsymbol{\sigma} \mapsto -\boldsymbol{\sigma}$ (a sign flip preserves rank, Frobenius norm and coherence), one may pick a maximiser $\hat{\boldsymbol{\sigma}}$ with $\langle \boldsymbol{Y}, \hat{\boldsymbol{\sigma}}\rangle_{\mathcal{T}} \geq 0$, reducing the joint MLE to $\hat{\boldsymbol{\sigma}} \in \arg\max_{\boldsymbol{\sigma}\in\mathcal{C}}\langle \boldsymbol{Y}, \boldsymbol{\sigma}\rangle_{\mathcal{T}}$ with $\hat{\lambda} = \langle \boldsymbol{Y}, \hat{\boldsymbol{\sigma}}\rangle_{\mathcal{T}} \geq 0$. ☐

**Lemma 8** (Tube Method)**.** *Let* $\Gamma_{R,\kappa} := \sup\{|\langle \boldsymbol{W}, \boldsymbol{\sigma}\rangle_{\mathcal{T}}| : \boldsymbol{\sigma} \in \mathfrak{S}_R,\ \kappa(\boldsymbol{\sigma}) \geq \kappa\}$*. Assume* $\boldsymbol{\sigma}^\star \in \mathfrak{S}_R$ *satisfies* $\kappa(\boldsymbol{\sigma}^\star) \geq \kappa$*. Then any estimator* $\hat{\boldsymbol{\sigma}} \in \mathfrak{S}_R$ *with* $\kappa(\hat{\boldsymbol{\sigma}}) \geq \kappa$ *satisfying* $\langle \boldsymbol{Y}, \hat{\boldsymbol{\sigma}}\rangle_{\mathcal{T}} \geq \langle \boldsymbol{Y}, \boldsymbol{\sigma}^\star\rangle_{\mathcal{T}}$ *obeys*

$$\|\hat{\boldsymbol{\sigma}} - \boldsymbol{\sigma}^\star\|_F^2 \;\leq\; \frac{4\,\Gamma_{R,\kappa}}{\lambda}\,, \qquad \textit{almost surely.} \tag{3.1}$$

*Proof of Lemma 8.* Substituting $\boldsymbol{Y} = \lambda\boldsymbol{\sigma}^\star + \boldsymbol{W}$ into $\langle \boldsymbol{Y}, \hat{\boldsymbol{\sigma}}\rangle_{\mathcal{T}} \geq \langle \boldsymbol{Y}, \boldsymbol{\sigma}^\star\rangle_{\mathcal{T}}$ gives

$$\lambda\langle \boldsymbol{\sigma}^\star, \hat{\boldsymbol{\sigma}}\rangle_{\mathcal{T}} + \langle \boldsymbol{W}, \hat{\boldsymbol{\sigma}}\rangle_{\mathcal{T}} \;\geq\; \lambda\|\boldsymbol{\sigma}^\star\|_F^2 + \langle \boldsymbol{W}, \boldsymbol{\sigma}^\star\rangle_{\mathcal{T}}\,.$$

Using $\|\boldsymbol{\sigma}^\star\|_F = \|\hat{\boldsymbol{\sigma}}\|_F = 1$ together with $\|\hat{\boldsymbol{\sigma}} - \boldsymbol{\sigma}^\star\|_F^2 = 2(1 - \langle \hat{\boldsymbol{\sigma}}, \boldsymbol{\sigma}^\star\rangle_{\mathcal{T}})$,

$$\tfrac{\lambda}{2}\,\|\hat{\boldsymbol{\sigma}} - \boldsymbol{\sigma}^\star\|_F^2 \;\leq\; \langle \boldsymbol{W}, \hat{\boldsymbol{\sigma}} - \boldsymbol{\sigma}^\star\rangle_{\mathcal{T}} \;\leq\; |\langle \boldsymbol{W}, \hat{\boldsymbol{\sigma}}\rangle_{\mathcal{T}}| + |\langle \boldsymbol{W}, \boldsymbol{\sigma}^\star\rangle_{\mathcal{T}}| \;\leq\; 2\,\Gamma_{R,\kappa}\,,$$

where the last inequality uses $\kappa(\hat{\boldsymbol{\sigma}}), \kappa(\boldsymbol{\sigma}^\star) \geq \kappa$, so that both inner products are bounded by the supremum defining $\Gamma_{R,\kappa}$. Multiplying by $2/\lambda$ yields (3.1). ☐

**Remark 3** (Role of the coherence constraint)**.** *The restriction* $\kappa(\hat{\boldsymbol{\sigma}}) \geq \kappa$ *is essential. Without it, the noise supremum* $\sup\{|\langle \boldsymbol{W}, \boldsymbol{\sigma}\rangle_{\mathcal{T}}| : \boldsymbol{\sigma} \in \mathfrak{S}_R\}$ *diverges as the rank-R components* $t_j$ *become near-collinear: in that regime the coefficients* $a_j$ *in* (1.2a) *can be arbitrarily large in magnitude while keeping* $\|\boldsymbol{\sigma}\|_F^2 = 1$*, since the Gram matrix* $\boldsymbol{G}$ *becomes singular. Restricting to* $\kappa(\hat{\boldsymbol{\sigma}}) \geq \kappa$ *confines the analysis to configurations where* $\lambda_{\min}(\boldsymbol{G}) \geq \kappa^2$*, ensuring* $\|\boldsymbol{a}\|_2 \leq 1/\kappa$ *(Lemma 9, Step 1) and a non-trivial tail bound.*

**Lemma 9** (Rank reduction)**.** *For any tensor* $\boldsymbol{\tau} \in \mathcal{T}(k, d)$ *and any normalized rank-R tensor* $\boldsymbol{\sigma} \in \mathfrak{S}_R$*,*

$$|\langle \boldsymbol{\tau}, \boldsymbol{\sigma}\rangle_{\mathcal{T}}| \;\leq\; \frac{\sqrt{R}}{\kappa(\boldsymbol{\sigma})}\, \sup_{\boldsymbol{\theta}\in\mathbb{S}^{d-1}} |\langle \boldsymbol{\tau}, \boldsymbol{\theta}^{\otimes k}\rangle_{\mathcal{T}}|\,. \tag{3.2}$$

*In particular,* $\Gamma_{R,\kappa} \leq (\sqrt{R}/\kappa)\,\Gamma_{1,1}$ *with* $\Gamma_{1,1}$ *defined by* (1.4a)*.*

*Proof of Lemma 9.* Let $\boldsymbol{\sigma} = \sum_{j=1}^R a_j t_j^{\otimes k}$ be a decomposition attaining the maximum in (1.2b), and write $\boldsymbol{G}$ for the Gram matrix $G_{ij} = \langle t_i, t_j\rangle^k$, whose smallest eigenvalue is $\kappa^2(\boldsymbol{\sigma})$ by definition.

- The normalization $\|\boldsymbol{\sigma}\|_F^2 = 1$ expands as

$$1 \;=\; \Big\langle \sum_{i=1}^R a_i t_i^{\otimes k}, \sum_{j=1}^R a_j t_j^{\otimes k}\Big\rangle_{\mathcal{T}} \;=\; \sum_{i,j=1}^R a_i a_j \langle t_i, t_j\rangle^k \;=\; \boldsymbol{a}^\top \boldsymbol{G}\,\boldsymbol{a}\,.$$

Combined with the Rayleigh quotient inequality $\boldsymbol{a}^\top\boldsymbol{G}\,\boldsymbol{a} \geq \lambda_{\min}(\boldsymbol{G})\,\|\boldsymbol{a}\|_2^2 = \kappa^2(\boldsymbol{\sigma})\,\|\boldsymbol{a}\|_2^2$,

$$\|\boldsymbol{a}\|_2 \;\leq\; 1/\kappa(\boldsymbol{\sigma})\,.$$

- By Cauchy-Schwarz

$$|\langle \boldsymbol{\tau}, \boldsymbol{\sigma} \rangle_{\mathcal{T}}| \;=\; \Big| \sum_{j=1}^{R} a_j \, \langle \boldsymbol{\tau}, t_j^{\otimes k} \rangle_{\mathcal{T}} \Big| \;\le\; \|\boldsymbol{a}\|_2 \, \Big( \sum_{j=1}^{R} \langle \boldsymbol{\tau}, t_j^{\otimes k} \rangle_{\mathcal{T}}^2 \Big)^{1/2} .$$

- Each $t_j \in \mathbb{S}^{d-1}$, so

$$\Big( \sum_{j=1}^{R} \langle \boldsymbol{\tau}, t_j^{\otimes k} \rangle_{\mathcal{T}}^2 \Big)^{1/2} \;\le\; \sqrt{R} \sup_{\boldsymbol{\theta} \in \mathbb{S}^{d-1}} |\langle \boldsymbol{\tau}, \boldsymbol{\theta}^{\otimes k} \rangle_{\mathcal{T}}| \, .$$

Combining the three steps yields (3.2). □

## 3.2 Proof of the main Theorem

*Proof of Theorem 1.* By construction of the profile MLE (1.3), $\hat{\boldsymbol{\sigma}}$ satisfies $\langle \boldsymbol{Y}, \hat{\boldsymbol{\sigma}} \rangle_{\mathcal{T}} \ge \langle \boldsymbol{Y}, \boldsymbol{\sigma}^\star \rangle_{\mathcal{T}}$ and $\kappa(\hat{\boldsymbol{\sigma}}) \ge \kappa$, so Lemma 8 applies and yields $\|\hat{\boldsymbol{\sigma}} - \boldsymbol{\sigma}^\star\|_F^2 \le 4\Gamma_{R,\kappa}/\lambda$. Lemma 9 then gives $\Gamma_{R,\kappa} \le (\sqrt{R}/\kappa)\,\Gamma_{1,1}$, hence

$$\|\hat{\boldsymbol{\sigma}} - \boldsymbol{\sigma}^\star\|_F^2 \;\le\; \frac{4\sqrt{R}}{\kappa\,\lambda}\,\Gamma_{1,1} \, . \tag{3.3}$$

On the complementary event $\{\Gamma_{1,1} \le u\}$, substituting (3.3) immediately yields $\|\hat{\boldsymbol{\sigma}} - \boldsymbol{\sigma}^\star\|_F^2 \le 4\sqrt{R}\,u/(\kappa\lambda)$, which is (1.7b).

It remains to bound the probability of the bad event $\{\Gamma_{1,1} > u\}$. By the symmetry $X \overset{d}{=} -X$ of the Gaussian field, the two-sided supremum satisfies

$$\mathbb{P}\{\Gamma_{1,1} > u\} \;\le\; 2\,\mathbb{P}\Big\{ \sup_{\boldsymbol{\theta} \in \mathbb{S}^{d-1}} X(\boldsymbol{\theta}) > u \Big\} . \tag{3.4}$$

For $u \ge u_{\mathsf{IMF}}$, Theorem 3 bounds the right-hand side by $\delta_{\mathsf{IMF}}(u)$. Multiplying by 2 per the symmetry reduction (3.4) and the definition (1.7a) of $\delta_{\min}(u) := 2\,\delta_{\mathsf{IMF}}(u)$ gives $\mathbb{P}\{\Gamma_{1,1} > u\} \le \delta_{\min}(u)$, which completes the proof. The looser bounds $2\,\delta_{\mathsf{SMF}}(u)$ on $[u_{\mathsf{SMF}}, \infty)$ and $2\,\delta_{\mathsf{SM}}(u)$ on $[u_{\mathsf{SM}}, \infty)$ (from Theorems 5 and 6) are not needed for this proof, but appear in the comparison of Section 2.6. □

## 3.3 Information-theoretic baselines and the likelihood ratio test

The geometric error of the profile MLE can be benchmarked against the information-theoretic distances between the planted-signal and pure-noise distributions.

**Proposition 7** (KL divergence, $\chi^2$ divergence, and log-likelihood ratio)**.** *Let $\boldsymbol{\tau} = \lambda \boldsymbol{\sigma}^\star \in \mathcal{T}(k,d)$ with $\lambda > 0$ and $\boldsymbol{\sigma}^\star \in \mathbb{S}(k,d)$. Then*

$$\mathrm{KL}(\mathbb{P}_0 \,\|\, \mathbb{P}_{\boldsymbol{\tau}}) \;=\; \frac{\lambda^2}{2} , \qquad \chi^2(\mathbb{P}_{\boldsymbol{\tau}} \,\|\, \mathbb{P}_0) \;=\; e^{\lambda^2} - 1 ,$$

*and almost surely*

$$\log \frac{\mathrm{d}\mathbb{P}_{\boldsymbol{\tau}}}{\mathrm{d}\mathbb{P}_0}(\boldsymbol{Y}) \;=\; \lambda \, \langle \boldsymbol{Y}, \boldsymbol{\sigma}^\star \rangle_{\mathcal{T}} - \frac{\lambda^2}{2} . \tag{3.5}$$

*Proof of Proposition 7.* The log-likelihood ratio reads

$$\log \frac{\mathrm{d}\mathbb{P}_{\boldsymbol{\tau}}}{\mathrm{d}\mathbb{P}_0}(\boldsymbol{Y}) \;=\; \tfrac{1}{2}\|\boldsymbol{Y}\|_F^2 - \tfrac{1}{2}\|\boldsymbol{Y} - \boldsymbol{\tau}\|_F^2 \;=\; \langle \boldsymbol{Y}, \boldsymbol{\tau} \rangle_{\mathcal{T}} - \tfrac{1}{2}\|\boldsymbol{\tau}\|_F^2 \;=\; \lambda \langle \boldsymbol{Y}, \boldsymbol{\sigma}^\star \rangle_{\mathcal{T}} - \tfrac{\lambda^2}{2} ,$$

since $\|\boldsymbol{\tau}\|_F = \lambda$ and $\boldsymbol{\tau} = \lambda \boldsymbol{\sigma}^\star$. Under $\mathbb{P}_0$ one has $\mathbb{E}_{\mathbb{P}_0}[\langle \boldsymbol{Y}, \boldsymbol{\sigma}^\star \rangle_{\mathcal{T}}] = \mathbb{E}_{\mathbb{P}_0}[\langle \boldsymbol{W}, \boldsymbol{\sigma}^\star \rangle_{\mathcal{T}}] = 0$ (as $\mathbb{E}[\boldsymbol{W}] = 0$), so $\mathbb{E}_{\mathbb{P}_0}\big[\log \frac{\mathrm{d}\mathbb{P}_{\boldsymbol{\tau}}}{\mathrm{d}\mathbb{P}_0}\big] = -\frac{\lambda^2}{2}$; hence $\mathrm{KL}(\mathbb{P}_0 \,\|\, \mathbb{P}_{\boldsymbol{\tau}}) = -\mathbb{E}_{\mathbb{P}_0}\big[\log \frac{\mathrm{d}\mathbb{P}_{\boldsymbol{\tau}}}{\mathrm{d}\mathbb{P}_0}\big] = \frac{\lambda^2}{2}$. For the $\chi^2$ divergence, the Gaussian moment generating function $\mathbb{E}_{\mathbb{P}_0} \exp(\langle \boldsymbol{W}, \boldsymbol{A} \rangle_{\mathcal{T}}) = \exp(\frac{1}{2}\|\boldsymbol{A}\|_F^2)$ gives

$$\mathbb{E}_{\mathbb{P}_0}\Big[\Big(\tfrac{\mathrm{d}\mathbb{P}_{\boldsymbol{\tau}}}{\mathrm{d}\mathbb{P}_0}(\boldsymbol{W})\Big)^2\Big] \;=\; e^{-\lambda^2}\, \mathbb{E}_{\mathbb{P}_0} \exp(2\lambda \langle \boldsymbol{W}, \boldsymbol{\sigma}^\star \rangle_{\mathcal{T}}) \;=\; e^{-\lambda^2} e^{2\lambda^2} \;=\; e^{\lambda^2} ,$$

hence $\chi^2(\mathbb{P}_{\boldsymbol{\tau}} \,\|\, \mathbb{P}_0) = e^{\lambda^2} - 1$. □

The likelihood ratio test (LRT) for the detection problem $H_0 : \lambda = 0$ versus $H_1 : \lambda > 0$ amounts, by (3.5), to thresholding the test statistic $\hat{\lambda}_{\mathsf{LRT}} = \max_{\boldsymbol{\sigma}\in\mathcal{C}} \langle \boldsymbol{Y}, \boldsymbol{\sigma}\rangle_{\mathcal{T}}$. Under $H_0$ one has $\boldsymbol{Y} = \boldsymbol{W}$, and since $\mathcal{C}$ is closed under $\boldsymbol{\sigma} \mapsto -\boldsymbol{\sigma}$ this statistic equals the (two-sided) noise supremum $\Gamma_{R,\kappa}$. The available tail control is on the *rank-one* field: $\delta_{\mathsf{min}}(u) = 2\delta_{\mathsf{IMF}}(u)$ bounds $\mathbb{P}\{\Gamma_{1,1} > u\}$, while the rank-reduction inequality (Lemma 9) only gives the deterministic, one-sided bound $\Gamma_{R,\kappa} \le (\sqrt{R}/\kappa)\,\Gamma_{1,1}$, with $\sqrt{R}/\kappa \ge 1$. Fix a target false-positive rate $\alpha \in (0, 1)$ and let $u_\alpha$ be the smallest level $u \ge u_{\mathsf{IMF}}$ with

$$\delta_{\mathsf{min}}(u_\alpha) \;\le\; \alpha\,. \tag{3.6}$$

The induced LRT rejects $H_0$ when $\hat{\lambda}_{\mathsf{LRT}} > (\sqrt{R}/\kappa)\, u_\alpha$, equivalently when the rescaled statistic $(\kappa/\sqrt{R})\,\hat{\lambda}_{\mathsf{LRT}}$ exceeds $u_\alpha$; its Type I error is then controlled at level $\alpha$,

$$\mathbb{P}_{H_0}\Big\{\hat{\lambda}_{\mathsf{LRT}} > \tfrac{\sqrt{R}}{\kappa}\, u_\alpha\Big\} = \mathbb{P}\Big\{\Gamma_{R,\kappa} > \tfrac{\sqrt{R}}{\kappa}\, u_\alpha\Big\} \le \mathbb{P}\{\Gamma_{1,1} > u_\alpha\} \le \delta_{\mathsf{min}}(u_\alpha) \le \alpha\,.$$

The inflation factor $\sqrt{R}/\kappa$ is exactly the one carried by the estimation bound (1.7b); for rank-one detection ($R = 1, \kappa = 1$) it equals 1. The non-asymptotic rate of Remark 1 gives $u_\alpha = O(\sqrt{d \log k + \log(1/\alpha)})$, hence a critical value $(\sqrt{R}/\kappa)\,O(\sqrt{d \log k + \log(1/\alpha)})$. Power against the alternative is governed by the recovery threshold $\lambda \gg \sqrt{d}$.

**Remark 4** (Numerical inversion of the tail bound)**.** *The master bound $\delta_{\mathsf{min}}(u) = 2\,\delta_{\mathsf{IMF}}(u)$ of Theorem 1 is, throughout its validity range $[u_{\mathsf{IMF}}, \infty)$, exactly twice the improved Mehta–Fyodorov bound $\delta_{\mathsf{IMF}}(u)$ of Theorem 3. This function admits a closed-form expression involving the Hermite partial sum $\Phi_d(\rho, u)$ and the rational function $\Psi_d(\rho, u)$ and is readily evaluated numerically; it is moreover strictly decreasing on $[u_{\mathsf{IMF}}, \infty)$, as is immediate from the Gaussian factors $e^{-u^2/2}$ and $e^{-(1+\rho^2)u^2/2}$ that dominate the polynomial prefactors. Consequently, for any prescribed significance level $\alpha \in (0, 1)$ one may apply a standard bisection method to (3.6) and obtain, in a few iterations, a numerically sharp estimate of the critical threshold $u_\alpha$ satisfying $\delta_{\mathsf{min}}(u_\alpha) = \alpha$.*

*This estimate is quantitatively sharp because the IMF bound itself is close to the true Kac–Rice probability: the Monte Carlo experiments in the companion repository `github.com/ydecastro/TENSOR-KSS` (*cf. *Figure 1) report a relative discrepancy between $\delta_{\mathsf{IMF}}(u)$ and the Monte Carlo estimate of the Kac–Rice integral $C_{k,d} \int_u^\infty \mathbb{E}[|\det(G_{d-1} - \rho x\, I)|]\, \varphi(x)\,\mathrm{d}x$ of at most $\sim 10^{-2}$ over the tested range of $(k, d)$ and $u$, so that the threshold obtained by inverting $\delta_{\mathsf{IMF}}$ differs from the exact Kac–Rice threshold by at most a fraction of a decimal place at the practical confidence levels $\alpha \in \{10^{-3}, 5 \times 10^{-2}\}$ documented in Figure 2.*

*When ultimate precision is needed, one may alternatively invert the exact Kac–Rice expression (2.1b) itself: the integrand $\mathbb{E}[|\det(G_{d-1} - \rho x\, I)|]\, \varphi(x)$ is estimated by Monte Carlo sampling of* GOE$(d-1)$ *in the Mehta convention, quadrature yields a strictly decreasing function of $u$, and a second bisection returns the exact threshold $u_\alpha^{\mathsf{exact}}$. The super-exponential accuracy of the Kac–Rice representation guarantees that $u_\alpha^{\mathsf{exact}}$ is the statistically correct threshold for the likelihood ratio test. We implemented both routines and confirm that the discrepancy $|u_\alpha - u_\alpha^{\mathsf{exact}}|$ is negligible for practical significance levels.*

Figure 2 illustrates the inversion routine of Remark 4 on the pairs $(k, d) \in \{(3, 5), (4, 6)\}$ at the two levels $\alpha \in \{10^{-3}, 5 \times 10^{-2}\}$. In each panel, the two-sided curves $u \mapsto \delta_{\mathsf{min}}(u) = 2\delta_{\mathsf{IMF}}(u)$ and $u \mapsto 2\delta_{\mathsf{exact}}(u)$ are plotted on logarithmic scale and intersected with the horizontal line at height $\alpha$; the two vertical dotted lines report the thresholds $u_\alpha$ (bisection on $\delta_{\mathsf{min}} = 2\delta_{\mathsf{IMF}}$) and $u_\alpha^{\mathsf{exact}}$ (bisection on $2\delta_{\mathsf{exact}}$). Three points stand out. *First*, at the stringent level $\alpha = 10^{-3}$ the two thresholds agree to within $|u_\alpha - u_\alpha^{\mathsf{exact}}| \le 3.4 \times 10^{-2}$ in every tested configuration, so the IMF bound is tight enough to set the LRT threshold at low false-positive rates. *Second*, at the loose level $\alpha = 5 \times 10^{-2}$ the agreement degrades to $|u_\alpha - u_\alpha^{\mathsf{exact}}| \approx 9 \times 10^{-2}$, reflecting the fact that the Hermite and Szegő bounds of Lemma 5 are increasingly slack as $u$ descends toward $u_{\mathsf{IMF}}$, yet the theoretical threshold $u_\alpha$ remains a strict upper bound on $u_\alpha^{\mathsf{exact}}$, producing a test with actual Type I error at most $\alpha$. *Third*, the gap widens with $d$: the super-exponential prefactor $\alpha_d \sim (e/(2d))^{d/2}$ shrinks faster than its numerical surrogate, so the theoretical inversion progressively over-estimates the threshold, which is exactly the *safe* direction for a statistical test.

**Remark 5** (Sharpness of the validity threshold $u_{\mathsf{IMF}}$)**.** *Figure 2 plots $\delta_{\mathsf{IMF}}$ and $\delta_{\mathsf{exact}}$ on $[u_{\mathsf{IMF}}, \infty)$, where Theorem 3 certifies $\delta_{\mathsf{exact}} \le \delta_{\mathsf{IMF}}$ via the discarding argument of Lemma 3. Below $u_{\mathsf{IMF}}$ the discarding*

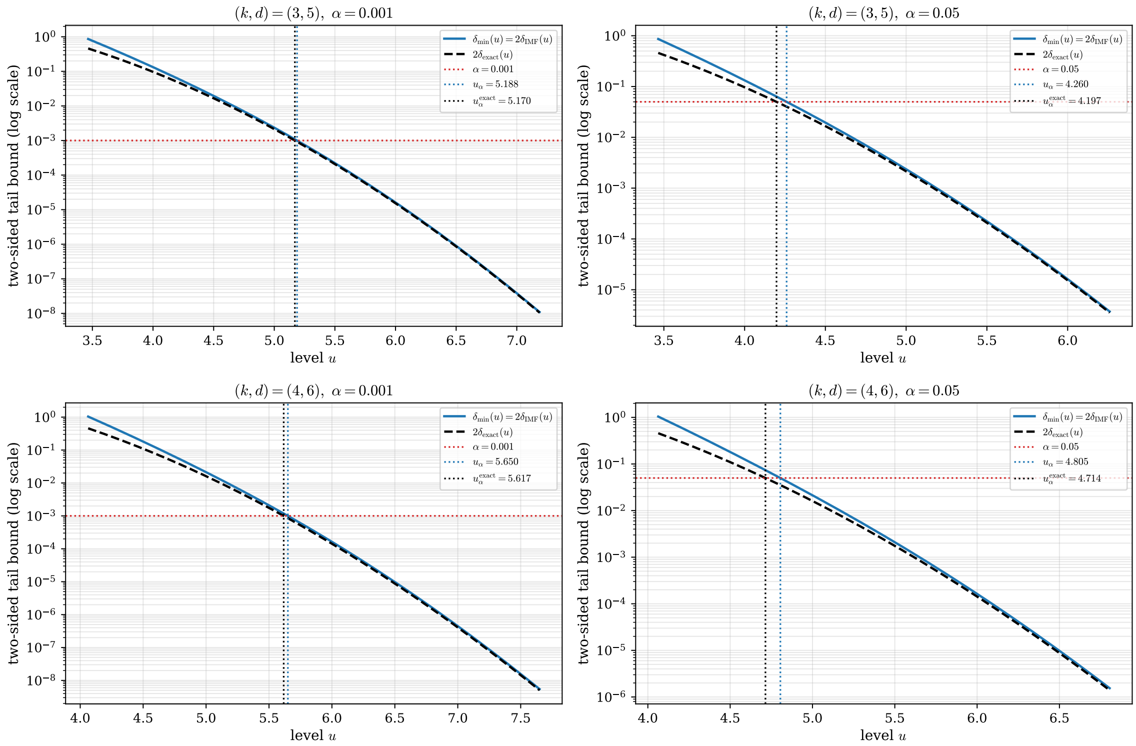


Figure 2: Numerical inversion of the *two-sided* master tail bound at significance levels $\alpha \in \{10^{-3}, 5\times 10^{-2}\}$ for $(k,d) \in \{(3,5),(4,6)\}$. In each panel: the two-sided IMF bound $\delta_{\min}(u) = 2\delta_{\mathrm{IMF}}(u)$ (blue), the two-sided exact bound $2\delta_{\mathrm{exact}}(u)$ of Theorem 2 (black dashed), the horizontal target level $\alpha$ (red dotted), the threshold $u_\alpha$ obtained by bisecting $\delta_{\min}(u)-\alpha$ (blue dotted vertical), and the exact threshold $u_\alpha^{\mathrm{exact}}$ obtained by bisecting $2\delta_{\mathrm{exact}}(u) - \alpha$ (black dotted vertical). The two-sided $\delta_{\min}$ is the master bound of (1.7a) that governs the Type I error of the LRT, whose statistic is the two-sided supremum $\Gamma_{R,\kappa}$. Since $\delta_{\mathrm{exact}} \le \delta_{\mathrm{IMF}}$ one has $u_\alpha^{\mathrm{exact}} \le u_\alpha$, so the IMF threshold is a conservative surrogate: here $u_\alpha = 5.188$, $4.260$ for $(3,5)$ and $5.650$, $4.805$ for $(4,6)$, against $u_\alpha^{\mathrm{exact}} = 5.170$, $4.197$ and $5.617$, $4.714$ respectively (the gap widens mildly as $\alpha$ grows and as $d$ increases). The induced LRT rejects $H_0$ when $\hat{\lambda}_{\mathrm{LRT}} > (\sqrt{R}/\kappa)\, u_\alpha$ (the rank-reduction inflation of Lemma 9; for rank-one detection $R=\kappa=1$ the factor is 1), guaranteeing Type I error at most $\alpha$.

*argument of Lemma 3 fails (the positivity of $\mathcal{I}_d^c(\rho x)$ on $[u_{\mathrm{IMF}},\infty)$ is not guaranteed), so $\delta_{\mathrm{IMF}}(u)$ ceases to be an upper bound on the Kac–Rice integral. An exploratory numerical evaluation (not displayed) suggests that $u_{\mathrm{IMF}}$ is moreover sharp in the following empirical sense: in every tested configuration, $\delta_{\mathrm{IMF}}(u)$ drops strictly below $\delta_{\mathrm{exact}}(u)$ on a window immediately to the left of $u_{\mathrm{IMF}}$, before rising again at very small $u$ via the rational $1/u$ contribution in $\Phi_d$ for $d$ odd.*

# 4 A non-asymptotic annealed complexity

The Kostlan–Shub–Smale field $X(\boldsymbol{\theta}) = \langle \boldsymbol{W}, \boldsymbol{\theta}^{\otimes k}\rangle_{\mathcal{T}}$ on $\mathbb{S}^{d-1}$ coincides with the Hamiltonian of the (centred, isotropic) spherical $k$-spin model from spin-glass physics (Auffinger et al., 2013; Ben Arous et al., 2019). Its critical-point structure, the count of local maxima above an energy level $E$, is a central object of study. Auffinger et al. (2013) write $\Theta_p$ for the *total* critical-point growth rate of the order-$p$ model and $\Theta_{0,p}$ for the local-maximum (index-0) rate, with $\Theta_{0,p}(u) = \Theta_p(u)$ above the spectral edge (that is, for $u \le -E_\infty$; their Eqs. (2.14),(2.16)). We write $\Theta_k := \Theta_{0,p}\big|_{p=k}$ for the local-maximum rate of the $k$-spin field, so that $\lim_{d\to\infty} d^{-1} \log \mathbb{E}[N^{\mathrm{lm}}_{[E,\infty)}] = \Theta_k(-u_{\mathrm{ABC}})$, where $u_{\mathrm{ABC}} = E/\sqrt{d}$ in our normalisation (the ABC energy scale $H/N$ with ambient dimension $N=d$; see Step 3(b) of the proof) and $N^{\mathrm{lm}}_{[E,\infty)}$ counts local maxima with

$X(\boldsymbol{\theta}) \geq E$. Here $E_\infty := 2\sqrt{(k-1)/k}$ is the ABC spectral-edge energy (Auffinger et al., 2013, Eq. (2.13)) (above which critical points are local maxima with overwhelming probability; equivalently the right edge of the limiting GOE spectral support, with $\rho E_\infty = \sqrt{2}$), and $E_0 > E_\infty$ is the ground-state energy above which no critical points survive (Auffinger et al., 2013, Theorem 2.5). Our four-tier hierarchy turns this asymptotic statement into a non-asymptotic two-sided bracketing at every finite $(k, d)$.

**Theorem 8** (Non-asymptotic annealed complexity)**.** *Let $k, d \geq 3$, let $X$ be the canonical Kostlan–Shub–Smale field on $\mathbb{S}^{d-1}$, and let $N^{\mathrm{lm}}_{[E,\infty)}$ denote the number of local maxima of $X$ with value at least $E$. Set*

$$E_{\mathrm{BDG}} := \frac{8\sqrt{2(d-1)}}{\rho}, \qquad C_{\mathrm{amp}} := 8\sqrt{2}, \qquad \delta_{\mathrm{BDG}}(\rho E) := e^{-2(\rho E)^2/9}. \tag{4.1a}$$

(i) (Upper bound.) *For every $E \in \mathbb{R}$,*

$$\mathbb{E}\big[N^{\mathrm{lm}}_{[E,\infty)}\big] \leq \mathbb{E}\big[N^{\mathrm{cp}}_{[E,\infty)}\big] = \delta_{\mathrm{exact}}(E), \tag{4.1b}$$

*where $N^{\mathrm{cp}}_{[E,\infty)}$ counts* all *critical points (not only local maxima) with $X(\boldsymbol{\theta}) \geq E$.*

(ii) (Two-sided bracketing, high-energy regime.) *For every $E \geq E_{\mathrm{BDG}}$,*

$$\big(1 - C_{\mathrm{amp}}^{d-1}\,\delta_{\mathrm{BDG}}^{1/2}(\rho E)\big)\,\delta_{\mathrm{exact}}(E) \leq \mathbb{E}\big[N^{\mathrm{lm}}_{[E,\infty)}\big] \leq \delta_{\mathrm{exact}}(E). \tag{4.1c}$$

**Corollary 8** (Annealed complexity of very deep local minima, by sign symmetry)**.** *Let $N^{\mathrm{lm,min}}_{(-\infty,E']}$ denote the number of local minima of the Kostlan–Shub–Smale field $X$ on $\mathbb{S}^{d-1}$ with value at most $E'$. The Gaussian symmetry $X \overset{d}{=} -X$ gives the distributional identity $N^{\mathrm{lm,min}}_{(-\infty,E']} \overset{d}{=} N^{\mathrm{lm}}_{[-E',\infty)}$ for every $E' \in \mathbb{R}$, and Theorem 8 transfers verbatim:*

(i′) (Upper bound.) *For every $E' \in \mathbb{R}$,*

$$\mathbb{E}\big[N^{\mathrm{lm,min}}_{(-\infty,E']}\big] \leq \delta_{\mathrm{exact}}(-E'). \tag{4.2a}$$

(ii′) (Two-sided bracketing, very-deep-minima regime.) *For every $E' \leq -E_{\mathrm{BDG}}$ (equivalently, $|E'| \geq 8\sqrt{2(d-1)}/\rho$),*

$$\big(1 - C_{\mathrm{amp}}^{d-1}\,\delta_{\mathrm{BDG}}^{1/2}(-\rho E')\big)\,\delta_{\mathrm{exact}}(-E') \leq \mathbb{E}\big[N^{\mathrm{lm,min}}_{(-\infty,E']}\big] \leq \delta_{\mathrm{exact}}(-E'). \tag{4.2b}$$

*The validity range $E' \leq -E_{\mathrm{BDG}}$ corresponds asymptotically to the reduced energy $E'/\sqrt{d} \leq -8E_\infty$: the bracket covers* very deep *local minima, far below the ground-state band $[-E_0, -E_\infty]$ identified by Auffinger et al. (2013, Theorem 2.5). Local minima in the moderate-energy band $E'/\sqrt{d} \in [-8E_\infty, -E_\infty]$ (where Auffinger et al. (2013, Theorem 2.4) provides the case-1 log-rate $\Theta_{0,p}$) and in the bulk band $E'/\sqrt{d} \in [-E_\infty, 0]$ (where ABC give the constant log-rate $\frac{1}{2}\log(p-1) - (p-2)/p$) are outside the scope of the BDG-amplifier framework developed in this paper; the upper bound* (i′) *remains unconditional throughout, however.*

**Remark 6** (Exact Kac–Rice representation)**.** *The Kac–Rice formula combined with the conditional shifted-GOE law of the Hessian (Auffinger et al., 2013, Lemma 3.2(b)) yields, for every $E \in \mathbb{R}$,*

$$\mathbb{E}\big[N^{\mathrm{lm}}_{[E,\infty)}\big] = C_{k,d}\int_E^\infty \mathbb{E}\big[|\det(G_{d-1} - \rho x\, I_{d-1})|\,\mathbb{1}\{G_{d-1} - \rho x I \prec 0\}\big]\,\varphi(x)\,\mathrm{d}x, \tag{4.3}$$

*with $C_{k,d} = 2\sqrt{\pi}(k-1)^{(d-1)/2}/\Gamma(d/2)$ and $\varphi$ the standard Gaussian density. The upper bound (4.1b) is obtained by dropping the indicator and applying Proposition 1:*

$$\mathbb{E}[N^{\mathrm{lm}}_{[E,\infty)}] \leq \mathbb{E}[\#\{\textit{critical points of } X \textit{ above } E\}] = \delta_{\mathrm{exact}}(E).$$

**Corollary 9** (Asymptotic match with Auffinger et al. (2013) in the high-energy regime)**.** *Recall $E_\infty$, $E_0$, and the local-maximum rate $\Theta_k$ from the start of this section: exponentially many local maxima persist throughout the band $E_\infty < E/\sqrt{d} < E_0$, where $\Theta_k(-eE_\infty) > 0$. The ABC complexity function (Auffinger et al., 2013, Theorem 2.4 and Eq. (2.15)) is given for $u \le -E_\infty$ by*

$$\Theta_k(u) \;=\; \tfrac{1}{2}\log(k-1) \;-\; \tfrac{k-2}{4(k-1)}u^2 \;-\; I_1(u)\,, \qquad I_1(u) := \tfrac{2}{E_\infty^2}\int_u^{-E_\infty} \sqrt{z^2 - E_\infty^2}\,\mathrm{d}z\,. \tag{4.4a}$$

*As $d \to \infty$ with $E = e\sqrt{2(d-1)}/\rho$ for fixed reduced energy $e \ge 1$, the level correspondence $E/\sqrt{d} \to eE_\infty$ holds, and the exact Kac–Rice equality $\mathbb{E}[N^{\mathrm{cp}}_{[E,\infty)}] = \delta_{\mathrm{exact}}(E)$ of part* (i) *together with Auffinger et al. (2013, Theorem 2.4) gives*

$$\lim_{d\to\infty} \frac{1}{d} \log \mathbb{E}\big[N^{\mathrm{lm}}_{[E,\infty)}\big] \;=\; \Theta_k(-eE_\infty)\,, \tag{4.4b}$$

*which, after the substitution $z = -E_\infty w$ in $I_1$, yields the explicit closed form*

$$\Theta_k(-eE_\infty) \;=\; \tfrac{1}{2}\log(k-1) \;-\; \tfrac{(k-2)e^2}{k} \;-\; 2\int_1^e \sqrt{w^2-1}\,\mathrm{d}w\,. \tag{4.4c}$$

*Proof.* Deferred to Appendix A. □

The threshold $e \ge 8$ governs only the *non-asymptotic two-sided bracket* (4.1c) – the validity range of the BDG amplifier in Theorem 8(ii) ($E \ge E_{\mathrm{BDG}} = 8\sqrt{2(d-1)}/\rho$). The asymptotic identity (4.4b) itself holds on the full ABC range $e \ge 1$, i.e. $u_{\mathrm{ABC}} \ge E_\infty$ for maxima (equivalently $u_{\mathrm{ABC}} \le -E_\infty$ for minima after sign symmetry), the case-1 branch of $\Theta_k$, since it follows from the exact equality of part (i) and Auffinger et al. (2013, Theorem 2.4) rather than from the bracket. The gap $e \in [1, 8)$ – moderate maxima above the spectral edge but below the BDG-amplifier threshold – is therefore still covered by the asymptotic (4.4b) (via the GOE semicircle confinement of Anderson et al. (2010, Theorem 2.1.22)) and by the unconditional upper bound (4.1b), but lies *outside the reach of the non-asymptotic two-sided bracket*: the BDG amplifier collapses below $e = 8$, and a matching finite-$d$ *lower* bracket on $e \in [1, 8)$ would require finer control of the largest eigenvalue of the GOE near the spectral edge (e.g. the LDP of Ben Arous et al. (2001, Theorem 6.2)). We leave that moderate-energy lower bracket as an open problem; in that range the asymptotic (4.4b) and the upper bound (4.1b) – which holds for every $E \in \mathbb{R}$ – remain in force.

**Remark 7** (Normalisation reconciliation with Auffinger et al. (2013))**.** *Auffinger et al. (2013) use $H_{N,p}$ on the sphere of radius $\sqrt{N}$ ($\mathbb{E}[H^2] = N$) with energy $u = H/N$; we use $f = H(\sqrt{N}\,\cdot)/\sqrt{N}$ on the unit sphere ($\mathbb{E}[f^2] = 1$) with energy $E$ on the scale $\sqrt{N}\,u$. The reduced energy satisfies $e = u_{\mathrm{ABC}}/E_\infty$ by* (A.5c)*, identifying the two conventions without further rescaling.*

**Remark 8** (Four nested non-asymptotic complexity bounds)**.** *Substituting $\delta_{\mathrm{IMF}}, \delta_{\mathrm{SMF}}, \delta_{\mathrm{SM}}$ for $\delta_{\mathrm{exact}}$ in* (4.1c) *yields four nested non-asymptotic bounds on the annealed complexity: $\delta_{\mathrm{exact}}$ tightest (Theorem 2), $\delta_{\mathrm{IMF}}$ asymptotically sharp, $\delta_{\mathrm{SMF}}$ inversion-friendly for complexity-vs-energy curves (Corollary 3), and $\delta_{\mathrm{SM}}$ independent of the Mehta–Fyodorov algebra.*

**Remark 9** (Spiked landscape: connection to the BAMMN result)**.** *Ben Arous et al. (2019) extend the complexity to the rank-one spiked model, where the conditional Hessian becomes the rank-one deformation $\widetilde{G}_{d-1} = \theta(m)\, e_1 e_1^\top + G_{d-1} - t(m,x) I_{d-1}$ with $\theta(m) = \sqrt{2k(k-1)}\,\lambda m^{k-2}(1-m^2)$, $t(m,x) = \sqrt{2k/(k-1)}\,x$ (Ben Arous et al., 2019, Eq. (4.42)); our unspiked result* (4.4b) *is its $\lambda = 0$ marginal ($\theta(m) = 0$). The* asymptotic *spiked complexity is well understood: Piccolo (2023) establishes the large-$d$ annealed complexity for an arbitrary finite number $r$ of spikes of mixed degrees, replacing the rank-one large deviation of Maïda (2007) by the finite-rank spherical-integral asymptotics of Guionnet and Husson (2022). Its $\lambda = 0$ specialisation coincides with* (4.4b) *after the noise-normalisation change of variables $u = \sqrt{2}\,x$ (both equal the Auffinger–Ben Arous–Černý rate, as we checked numerically). Our framework leaves open the* non-asymptotic, *finite-$(k, d)$ spiked bracket. This would require redoing the Mehta–Fyodorov algebra for the rank-one-deformed ensemble, which has no closed Fyodorov representation (Ben Arous et al., 2019, Lemma 4.4), together with non-asymptotic counterparts of those large-deviation and spherical-integral inputs (compare Maïda, 2007, Theorem 1.1 and Guionnet and Maï da, 2005, Theorem 6). We leave that finite-$d$ refinement as a follow-up.*

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# A Deferred proofs

For readability the three longest proofs are collected here.

*Proof of Theorem 2.* We check each $D_i$ in turn; only $D_2$ requires care.

• $D_4$ *(squared Hermite).* This is exactly the content of Theorem 4: by the change of variable $y = \rho x$ and Corollary 11,

$$D_4(u) \;=\; T_d^{\text{exact}}(u) \;=\; \rho^{-1}\sum_{j=0}^{d-1} c_j^2\, K_j^{\beta}(\rho u)\,, \qquad \beta = (3k-2)/k\,.$$

• $D_1, D_3$ *(linear Hermite).* The change of variable $y = \rho x$ in $\mathcal{L}_d(u) = \int_u^\infty H_{d-1}(\rho x)\, e^{-x^2/2}\,\mathrm{d}x$ gives $\mathcal{L}_d(u) = \rho^{-1} J_{d-1}^{1/\rho^2}(\rho u)$, and Lemma 15 with $\beta = 1/\rho^2$, $\gamma = 2\rho^2$, $\theta = 2\rho^2 - 1 = \Lambda$ supplies the closed form. For $d$ odd, $D_3$ exhibits a Mills-ratio-type remainder $\bar\Phi(u\sqrt{\beta})$ (since $d-1$ is even and $J_0^{1/\rho^2}$ enters); for $d$ even, $D_1$ has no such remainder (since $d-1$ is odd).

• $D_2$*, cross-integral, polynomial part.* Apply Lemma 14 to $\mathcal{I}_d^c(\rho x)$:

$$\mathcal{I}_d^c(\rho x) \;=\; e^{-(\rho x)^2/2}\sum_{\ell=0}^{\lfloor (d-1)/2\rfloor} \frac{2^{\ell+1}(d-1)!!}{(d-2\ell-1)!!}\, H_{d-2\ell-1}(\rho x) \;+\; \mathbb{1}_{\{d\text{ even}\}}\, 2^{d/2}(d-1)!!\int_{\rho x}^\infty e^{-y^2/2}\,\mathrm{d}y\,.$$

Substituting into (2.7), the Hermite-series part becomes

$$\sum_\ell \frac{2^{\ell+1}(d-1)!!}{(d-2\ell-1)!!}\int_u^\infty H_{d-1}(\rho x)\, H_{d-2\ell-1}(\rho x)\, e^{-(1+\rho^2)x^2/2}\,\mathrm{d}x\,.$$

The classical product linearization $H_a(\rho x)\, H_b(\rho x) = \sum_{p=0}^{\min(a,b)} 2^p\, p!\, \binom{a}{p}\binom{b}{p}\, H_{a+b-2p}(\rho x)$, combined with $y = \rho x$ and Lemma 15 with $\beta = (3k-2)/k$, evaluates each integral to $\rho^{-1} J^{\beta}_{a+b-2p}(\rho u)$, producing the first line of (2.8c).

• $D_2$*, cross-integral, Fubini remainder ($d$ even only).* The remainder contribution to $\mathcal{C}_d(u)$ is, after substituting $\mathcal{I}_d^c$'s Gaussian-tail piece,

$$2^{d/2}(d-1)!!\,\mathcal{F}_d(u)\,, \qquad \mathcal{F}_d(u) \;:=\; \int_u^\infty H_{d-1}(\rho x)\left[\int_{\rho x}^\infty e^{-y^2/2}\,\mathrm{d}y\right] e^{-x^2/2}\,\mathrm{d}x\,.$$

The double integral is over $\{(x, y) : x \ge u,\; y \ge \rho x\}$. By Fubini, swapping the order, $y$ ranges over $[\rho u, \infty)$ and for each such $y$, $x$ ranges over $[u, y/\rho]$:

$$\mathcal{F}_d(u) \;=\; \int_{\rho u}^\infty e^{-y^2/2}\int_u^{y/\rho} H_{d-1}(\rho x)\, e^{-x^2/2}\,\mathrm{d}x\,\mathrm{d}y \;=\; \frac{1}{\rho}\int_{\rho u}^\infty e^{-y^2/2}\big[J_{d-1}^{1/\rho^2}(\rho u) - J_{d-1}^{1/\rho^2}(y)\big]\,\mathrm{d}y\,,$$

where the second equality applies the change of variable $z = \rho x$ to the inner integral and writes the difference of $J_{d-1}^{1/\rho^2}$ at the two endpoints. Splitting at the bracket gives

$$\mathcal{F}_d(u) \;=\; \underbrace{\frac{J_{d-1}^{1/\rho^2}(\rho u)}{\rho}\int_{\rho u}^\infty e^{-y^2/2}\,\mathrm{d}y}_{=:\,\mathcal{F}_d^{(1)}(u)} \;-\; \underbrace{\frac{1}{\rho}\int_{\rho u}^\infty e^{-y^2/2}\, J_{d-1}^{1/\rho^2}(y)\,\mathrm{d}y}_{=:\,\mathcal{F}_d^{(2)}(u)}\,. \tag{A.1}$$

*First piece* $\mathcal{F}_d^{(1)}$. Direct evaluation: $\int_{\rho u}^\infty e^{-y^2/2}\,\mathrm{d}y = \sqrt{2\pi}\,\bar\Phi(\rho u)$, so

$$\mathcal{F}_d^{(1)}(u) \;=\; \sqrt{2\pi}\,\bar\Phi(\rho u)\,\frac{J_{d-1}^{1/\rho^2}(\rho u)}{\rho}\,,$$

which is the first term of (2.8d).

*Second piece* $\mathcal{F}_d^{(2)}$. For $d$ even, $d-1$ is odd, so the iterated form (D.9d) of Lemma 15 applied with $\beta = 1/\rho^2$, $\gamma = 2/\beta = 2\rho^2$, $\theta = (2-\beta)/\beta = 2\rho^2 - 1 = \Lambda$ has $R_{d-1}^{1/\rho^2} = 0$ and reads

$$J_{d-1}^{1/\rho^2}(y) \;=\; 2\rho^2\, e^{-y^2/(2\rho^2)} \sum_{k=0}^{(d-2)/2} (2\Lambda)^k\, \frac{(d-2)!!}{(d-2k-2)!!}\, H_{d-2k-2}(y)\,.$$

Substituting into $\mathcal{F}_d^{(2)}(u)$ and pulling the constants $2\rho^2$ and the outer $1/\rho$ outside the sum,

$$\begin{aligned}
\mathcal{F}_d^{(2)}(u) \;&=\; \frac{1}{\rho}\int_{\rho u}^{\infty} e^{-y^2/2}\cdot 2\rho^2\, e^{-y^2/(2\rho^2)} \sum_{k=0}^{(d-2)/2} (2\Lambda)^k\, \frac{(d-2)!!}{(d-2k-2)!!}\, H_{d-2k-2}(y)\,\mathrm{d}y \\
&=\; \underbrace{\frac{2\rho^2}{\rho}}_{=2\rho} \sum_{k=0}^{(d-2)/2} (2\Lambda)^k\, \frac{(d-2)!!}{(d-2k-2)!!} \int_{\rho u}^{\infty} H_{d-2k-2}(y)\, e^{-(1+1/\rho^2)y^2/2}\,\mathrm{d}y \\
&=\; 2\rho \sum_{k=0}^{(d-2)/2} (2\Lambda)^k\, \frac{(d-2)!!}{(d-2k-2)!!}\, J_{d-2k-2}^{\beta}(\rho u)\,,
\end{aligned}$$

where the combined exponent $-(1/\rho^2+1)y^2/2 = -\beta y^2/2$ identifies $\beta = 1 + 1/\rho^2 = (1+\rho^2)/\rho^2 = (3k-2)/k$, the same $\beta$ appearing in $D_4$ (Corollary 11), and the last line applies the definition $J_m^{\beta}(\rho u) = \int_{\rho u}^{\infty} H_m(y)\, e^{-\beta y^2/2}\,\mathrm{d}y$.

*Assembly.* Substituting $\mathcal{F}_d^{(1)}$ and $\mathcal{F}_d^{(2)}$ into (A.1) and noting the explicit minus sign in front of $\mathcal{F}_d^{(2)}$, the second-piece coefficient becomes $-2\rho$, yielding the second line of (2.8d).

• *Pointwise tightness.* The chain $\delta_{\text{exact}} \le \delta_{\text{IMF}}^{\star} \le \delta_{\text{IMF}}$ follows from the chain of relaxations: $\delta_{\text{IMF}}^{\star}$ discards $-2\mathcal{I}_d^c\, H_{d-1} \le 0$ on $\{\rho x \ge \sqrt{2d-1}\}$, and $\delta_{\text{IMF}}$ further replaces each $H_j(\rho x) \le (2\rho x)^j$ via Lemma 4. Both inequalities are strict at every finite $u$. $\square$

*Proof of Lemma 5.* • We separate the Mehta expansion (2.3) into three distinct pieces, $Q_d = T_1 + T_2 + T_3$, defined as follows:

$$\begin{aligned}
T_1(\nu) &= e^{-\nu^2/2} \sum_{j=0}^{d-1} c_j^2 H_j^2(\nu)\,, \\
T_2(\nu) &= \tfrac{1}{2}\sqrt{d/2}\, c_{d-1} c_d\, H_{d-1}(\nu)\left[\mu_d - 2\mathcal{I}_d^c(\nu)\right], \\
T_3(\nu) &= \mathbb{1}_{\{d \text{ odd}\}}\, \frac{H_{d-1}(\nu)}{\mu_{d-1}}\,.
\end{aligned}$$

• We extract the constant prefactor multiplying $H_{d-1}(\nu)$, which depends on the parity of the dimension $d$:

- **For $d$ even:** $T_3 = 0$, and the constant part of the bracket in $T_2$ isolates $\alpha_d = \frac{1}{2}\sqrt{d/2}\, c_{d-1} c_d\, \mu_d$.
- **For $d$ odd:** $\mu_d = 0$ (by parity), leaving only $T_3$ to contribute. This gives $\alpha_d = 1/\mu_{d-1}$ (here $d-1$ is even, so $\mu_{d-1} \neq 0$).

The exact values of the integrals are obtained via direct evaluation of the Gaussian moments

$$\int_{\mathbb{R}} y^{2j}\, e^{-y^2/2}\,\mathrm{d}y \;=\; \sqrt{2\pi}\,(2j-1)!!\,,$$

yielding the closed form $\mu_{2p} = \sqrt{2\pi}\,(2p)!/p!$. Applying Stirling's approximation produces the asymptotic equivalents of the lemma.

• The total remainder $\mathcal{R}_d(\nu) := Q_d(\nu) - \alpha_d H_{d-1}(\nu)$ has the same closed form in both parities:

- **For $d$ even**, $T_3 = 0$ and the constant part of $T_2$'s bracket is $\alpha_d H_{d-1}(\nu)$, so $\mathcal{R}_d = T_1 + (T_2 - \alpha_d H_{d-1}) = T_1 - \sqrt{d/2}\, c_{d-1} c_d\, \mathcal{I}_d^c(\nu)\, H_{d-1}(\nu)$.

- **For** $d$ **odd**, $\mu_d = 0$ so $T_2 = -\sqrt{d/2}\, c_{d-1} c_d\, \mathcal{I}^c_d(\nu)\, H_{d-1}(\nu)$, and $T_3 = H_{d-1}(\nu)/\mu_{d-1} = \alpha_d H_{d-1}(\nu)$, so $\mathcal{R}_d = T_1 + T_2 = T_1 - \sqrt{d/2}\, c_{d-1} c_d\, \mathcal{I}^c_d(\nu)\, H_{d-1}(\nu)$.

In both cases

$$\mathcal{R}_d(\nu) \;=\; T_1(\nu) \;-\; \sqrt{d/2}\, c_{d-1} c_d\, \mathcal{I}^c_d(\nu)\, H_{d-1}(\nu)\,, \tag{A.2}$$

and we bound the two terms separately.

We first bound $T_1$. The physicist Hermite polynomial admits the explicit expansion

$$H_j(\nu) \;=\; \sum_{m=0}^{\lfloor j/2 \rfloor} \frac{(-1)^m\, j!}{m!\,(j-2m)!} (2\nu)^{j-2m}\,,$$

and we use the elementary coefficient inequality $j!\, 2^{j-2m}/m! \le (2j)!/(2m)!$, valid for every $j \ge m \ge 0$. The latter is proved by induction: the base case $j = m$ gives $2^{-m} \le 1$, and the step $j \to j+1$ multiplies the LHS by $2(j+1)$ and the RHS by $2(2j+1)(j+1)$, giving a ratio of $2j+1 \ge 1$. The triangle inequality then yields

$$|H_j(\nu)| \;\le\; \sum_{m=0}^{\lfloor j/2 \rfloor} \frac{(2j)!}{(2m)!(j-2m)!} |\nu|^{j-2m} \;\le\; \sum_{p=0}^{j} \frac{(2j)!}{j!} \binom{j}{p} |\nu|^p \;=\; \frac{(2j)!}{j!} (1+|\nu|)^j\,,$$

where the second inequality adds the (non-negative) odd-power monomials $|\nu|^p$ with $p \not\equiv j \pmod 2$, using $(2j)!/((2m)!(j-2m)!) = (2j)!/j! \cdot j!/((2m)!(j-2m)!) \le (2j)!/j! \cdot \binom{j}{j-2m}$. Squaring and applying $(1+|\nu|)^2 \le 2(1+\nu^2)$,

$$T_1(\nu) \;\le\; S_d (1+\nu^2)^{d-1} e^{-\nu^2/2}\,, \qquad \text{where } S_d := \sum_{j=0}^{d-1} c_j^2\, 2^j \left[\frac{(2j)!}{j!}\right]^2.$$

• *A uniform envelope for* $\mathcal{I}^c_d$. We bound $\mathcal{I}^c_d$ on the whole half-line $\{\nu \ge 0\}$ by a single envelope carrying the same polynomial–Gaussian form $(1+|\nu|)^{d-1} e^{-\nu^2/2}$ as $H_{d-1}$. The explicit Hermite-tail series of Lemma 14 (applied with degree $m = d$) reads

$$\mathcal{I}^c_d(\nu) \;=\; e^{-\nu^2/2} \sum_{k=0}^{\lfloor (d-1)/2 \rfloor} 2^{k+1} \frac{(d-1)!!}{(d-2k-1)!!} H_{d-2k-1}(\nu) \;+\; R_d(\nu)\,,$$

with $R_d(\nu) = 0$ for $d$ odd and $R_d(\nu) = 2^{d/2}(d-1)!! \int_\nu^\infty e^{-y^2/2}\,\mathrm{d}y$ for $d$ even. By the coefficient bound $|H_m(\nu)| \le \frac{(2m)!}{m!}(1+|\nu|)^m$ established above for $T_1$ (whence $|H_{d-2k-1}(\nu)| \le \frac{(2(d-2k-1))!}{(d-2k-1)!}(1+|\nu|)^{d-2k-1} \le \frac{(2(d-2k-1))!}{(d-2k-1)!}(1+|\nu|)^{d-1}$ since $1+|\nu| \ge 1$) and the monotone-Mills bound $\int_\nu^\infty e^{-y^2/2}\,\mathrm{d}y \le \sqrt{\pi/2}\, e^{-\nu^2/2}$ (valid for every $\nu \ge 0$, because $\nu \mapsto e^{\nu^2/2} \int_\nu^\infty e^{-y^2/2}\,\mathrm{d}y$ is non-increasing, with value $\sqrt{\pi/2}$ at $\nu = 0$, since its derivative $\nu\, e^{\nu^2/2} \int_\nu^\infty e^{-y^2/2}\mathrm{d}y - 1$ is negative by the Mills inequality $\nu\, e^{\nu^2/2}\int_\nu^\infty e^{-y^2/2}\mathrm{d}y < 1$), we obtain

$$\begin{aligned} |\mathcal{I}^c_d(\nu)| \;&\le\; E_d\, (1+|\nu|)^{d-1}\, e^{-\nu^2/2}\,, \\ E_d \;&:=\; \sum_{k=0}^{\lfloor (d-1)/2 \rfloor} 2^{k+1} \frac{(d-1)!!}{(d-2k-1)!!} \frac{(2(d-2k-1))!}{(d-2k-1)!} \;+\; 2^{d/2}(d-1)!! \sqrt{\tfrac{\pi}{2}}\,. \end{aligned} \tag{A.3}$$

• $E_d$ *is dominated by* $\widetilde{B}_d$. The summands $a_k := 2^{k+1} \frac{(d-1)!!}{(d-2k-1)!!} \frac{(2(d-2k-1))!}{(d-2k-1)!}$ are strictly decreasing in $k$: with $p := d - 2k - 1 \ge 2$, a direct computation gives the ratio $a_{k+1}/a_k = \frac{p}{2(2p-1)(2p-3)} \le \frac{1}{2(2p-3)} < 1$. Hence, using $a_0 = 2\frac{(2d-2)!}{(d-1)!}$ and that the sum has at most $\lfloor (d-1)/2 \rfloor + 1 \le \frac{d+1}{2}$ terms,

$$\sum_k a_k \;\le\; \frac{d+1}{2}\, a_0 \;=\; (d+1)\frac{(2d-2)!}{(d-1)!}\,, \qquad 2^{d/2}(d-1)!!\sqrt{\tfrac{\pi}{2}} \;\le\; 2^d \frac{(2d-2)!}{(d-1)!}\,,$$

the second bound using $(d-1)!! \le (d-1)! \le \frac{(2d-2)!}{(d-1)!}$ and $\sqrt{\pi/2} \le 2^{d/2}$. Therefore

$$E_d \;\le\; (d+1+2^d)\frac{(2d-2)!}{(d-1)!} \;\le\; (2d-1)\, 2^{d+2}\, \frac{(2d-2)!}{(d-1)!} \;=\; \frac{(2d)!\, 2^{d+1}}{d!} \;\le\; \widetilde{B}_d\,,$$

where

$$\widetilde{B}_d := \max\left(\frac{(2d)!\,2^{d+1}}{d!},\ B'_d\right), \qquad B'_d := \frac{(2d)!}{d!}\int_0^\infty (1+y)^d e^{-y^2/2}\,\mathrm{d}y\,,$$

($B'_d$ is the constant from the crude global bound $|\mathcal{I}_d^c(\nu)| \le \int_0^\infty |H_d(y)|e^{-y^2/2}\,\mathrm{d}y = B'_d$, retained in the definition of $\widetilde{B}_d$ for definiteness; the envelope (A.3) already gives $E_d \le \widetilde{B}_d$ directly). Substituting into (A.3),

$$|\mathcal{I}_d^c(\nu)| \ \le\ \widetilde{B}_d\,(1+|\nu|)^{d-1}\,e^{-\nu^2/2} \qquad \text{for every } \nu \ge 0.$$

• *Assembly.* Combining the last bound with $|H_{d-1}(\nu)| \le \frac{(2d-2)!}{(d-1)!}(1+|\nu|)^{d-1}$ and the elementary inequality $(1+|\nu|)^{2(d-1)} \le 2^{d-1}(1+\nu^2)^{d-1}$ (which follows from $(1+|\nu|)^2 \le 2(1+\nu^2)$), the cross term in (A.2) obeys

$$\sqrt{d/2}\,c_{d-1}c_d\,|\mathcal{I}_d^c(\nu)|\,|H_{d-1}(\nu)| \ \le\ \sqrt{d/2}\,c_{d-1}c_d\,\frac{(2d-2)!}{(d-1)!}\,2^{d-1}\,\widetilde{B}_d\,(1+\nu^2)^{d-1}e^{-\nu^2/2}\,.$$

Adding the bound $T_1(\nu) \le S_d\,(1+\nu^2)^{d-1}e^{-\nu^2/2}$ established above and setting

$$\beta_d := S_d + \sqrt{d/2}\,c_{d-1}c_d\,\frac{(2d-2)!}{(d-1)!}\,2^{d-1}\,\tilde{B}_d$$

yields the remainder bound (2.13c). □

*Proof of Theorem 8.* The inputs are the Kac–Rice formula for smooth Gaussian fields on the sphere (Azaïs and Wschebor, 2009, Theorem 6.4), the conditional GOE law of the Hessian (Auffinger et al., 2013, Lemma 3.2(b)), and the Ben Arous–Dembo–Guionnet large-deviation bound on the GOE spectral radius (Lemma 13).

• *Step 1: Exact Kac–Rice representation and upper bound.* The Kac–Rice computation here is the one of Section 2 (displays (2.1a)–(2.1b) and Proposition 1), with two changes. First, the target is the expected *count* of local maxima with value in $[E,\infty)$, not a supremum probability: for this count the Kac–Rice formula (Azaïs and Wschebor, 2009, Theorem 6.4) is an *equality*, with the same critical-point intensity $\bar{p}(x)$ as in Section 2 and with the indicator $\mathbb{1}\{\nabla^2 X(\boldsymbol{\theta}) \prec 0\}$ kept inside the conditional expectation. Second, this indicator passes to the GOE side: by the conditional Hessian law (2.1a), $\nabla^2 X(\boldsymbol{\theta})$ given $X(\boldsymbol{\theta}) = x$ is the *positive* multiple $\sqrt{2k(k-1)}\,(G_{d-1} - \rho x\, I_{d-1})$ of a shifted GOE (Auffinger et al., 2013, Lemma 3.2(b)), so $\nabla^2 X(\boldsymbol{\theta}) \prec 0$ if and only if $G_{d-1} - \rho x\, I_{d-1} \prec 0$, equivalently $\max_i \mu_i < \rho x$, where $\mu_1 \le \cdots \le \mu_{d-1}$ are the eigenvalues of $G_{d-1}$. With these two changes, the constants gather exactly as in (2.1b) and give the exact representation (4.3).

Dropping the indicator $\mathbb{1}\{G_{d-1} - \rho x I \prec 0\} \le 1$ in (4.3) removes the only difference with the Kac–Rice integral of Section 2, so the same calculation as in Proposition 1 gives

$$\mathbb{E}[N^{\mathrm{lm}}_{[E,\infty)}] \ \le\ C_{k,d}\int_E^\infty \mathbb{E}\big[|\det(G_{d-1} - \rho x I)|\big]\,\varphi(x)\,\mathrm{d}x \ =\ 2(k-1)^{\frac{d-1}{2}}\int_E^\infty Q_d(\rho x)\,e^{-x^2/2}\,\mathrm{d}x \ =\ \delta_{\mathrm{exact}}(E)\,,$$

where the middle integral is the Kac–Rice expected number of *all* critical points of $X$ above level $E$, and the last equality is the integral form of $\delta_{\mathrm{exact}}$ in Theorem 2 (eq. (2.8a)). This proves part (i), with no symmetrisation factor: the bound on $\mathbb{P}\{\sup X > u\}$ delivered by $\delta_{\mathrm{exact}}$ is already one-sided, and the two-sided factor 2 of (3.4) appears explicitly in Theorem 1 (as $2\,\delta_{\mathrm{exact}} \le \delta_{\min}$), not inside $\delta_{\mathrm{exact}}$ itself.

• *Step 2: Lower bound, valid for $E \ge E_{\mathrm{BDG}}$, with explicit constants.* For the lower bound we exploit that on the event $\mathcal{E} := \{M_{d-1} \le \rho E\}$, where $M_{d-1} := \max_i |\mu_i|$, one has $|\mu_i| \le \rho E \le \rho x$ for every $x \ge E$, hence $\rho x - \mu_i > 0$ and a fortiori $G_{d-1} - \rho x I \prec 0$. Lemma 13 applies on $\rho E \ge 4\sqrt{2(d-1)}$, which is automatic under $E \ge E_{\mathrm{BDG}} = 8\sqrt{2(d-1)}/\rho$:

$$\mathbb{P}(\mathcal{E}^c) \ =\ \mathbb{P}(M_{d-1} > \rho E) \ \le\ e^{-2(\rho E)^2/9} \ =:\ \delta_{\mathrm{BDG}}(\rho E)\,.$$

On $\mathcal{E}$ the determinant satisfies $|\det(G - \rho x I)| = \prod_i(\rho x - \mu_i) \ge 0$, so

$$\mathbb{E}\big[|\det(G - \rho x I)|\,\mathbb{1}\{G - \rho x I \prec 0\}\big] \ \ge\ \mathbb{E}\big[|\det(G - \rho x I)|\,\mathbb{1}_{\mathcal{E}}\big] \ =\ \mathbb{E}\big[|\det(G - \rho x I)|\big] - \mathbb{E}\big[|\det(G - \rho x I)|\mathbb{1}_{\mathcal{E}^c}\big]\,.$$

We bound the remainder term, and then the denominator, with explicit constants.

*(4a) GOE moment bound:* $C_1 = 64$. By Lemma 13, $\mathbb{P}(M_{d-1} > t) \le e^{-2t^2/9}$ for $t \ge T_1 := 4\sqrt{2(d-1)}$. The layer-cake formula gives

$$\mathbb{E}[M_{d-1}^{2(d-1)}] \;=\; \int_0^\infty 2(d-1)\, t^{2d-3}\, \mathbb{P}(M_{d-1} > t)\, \mathrm{d}t \;\le\; T_1^{2(d-1)} \;+\; 2(d-1) \int_{T_1}^\infty t^{2d-3} e^{-2t^2/9}\, \mathrm{d}t\,.$$

The first term equals $(4\sqrt{2(d-1)})^{2(d-1)} = 32^{d-1}(d-1)^{d-1}$. For the second, Lemma 10 with $m = 2d-3$, $a = 4/9$ gives, on $T_1^2 = 32(d-1)$ and $(m-1)/T_1^2 = (2d-4)/(32(d-1)) \le 1/16$,

$$\int_{T_1}^\infty t^{2d-3} e^{-2t^2/9} \mathrm{d}t \;\le\; \frac{T_1^{2d-4}\, e^{-2T_1^2/9}}{4/9 - 1/16} \;=\; \frac{(32(d-1))^{d-2}\, e^{-64(d-1)/9}}{55/144}\,.$$

Multiplying by $2(d-1)$ and simplifying,

$$2(d-1) \int_{T_1}^\infty t^{2d-3} e^{-2t^2/9} \mathrm{d}t \;\le\; \frac{288}{55} \cdot \frac{(d-1)(32(d-1))^{d-2}}{1}\, e^{-64(d-1)/9} \;\le\; 32^{d-1}(d-1)^{d-1} \cdot e^{-1}$$

for $d \ge 2$, where the last step uses $(288/(55 \cdot 32)) \cdot e^{-64(d-1)/9+1} \le 1$. Adding the two pieces,

$$\mathbb{E}[M_{d-1}^{2(d-1)}] \;\le\; 32^{d-1}(d-1)^{d-1}\,(1 + e^{-1}) \;\le\; 64^{d-1}(d-1)^{d-1} \qquad \text{for every } d \ge 2. \tag{A.4a}$$

This pins $C_1 = 64$ explicitly.

*(4b) Numerator bound:* $C_2 = 8$. On $\mathcal{E}^c$, by the triangle inequality $|\rho x - \mu_i| \le \rho x + |\mu_i| \le \rho x + M_{d-1}$, so $|\det(G - \rho x I)| \le (\rho x + M_{d-1})^{d-1}$, and by Cauchy–Schwarz,

$$\mathbb{E}[(\rho x + M_{d-1})^{d-1} \mathbb{1}_{\mathcal{E}^c}] \;\le\; \sqrt{\mathbb{E}[(\rho x + M_{d-1})^{2(d-1)}]\, \mathbb{P}(\mathcal{E}^c)}\,.$$

Using $(\rho x + M_{d-1})^{2(d-1)} \le 4^{d-1}((\rho x)^{2(d-1)} + M_{d-1}^{2(d-1)})$ and (A.4a),

$$\mathbb{E}[(\rho x + M_{d-1})^{2(d-1)}] \;\le\; 4^{d-1}\big[(\rho x)^{2(d-1)} + 64^{d-1}(d-1)^{d-1}\big]\,.$$

On $\rho x \ge \rho E \ge 8\sqrt{2(d-1)}$, $(\rho x)^2 \ge 2 \cdot 64(d-1)$, hence $64^{d-1}(d-1)^{d-1} \le (\rho x)^{2(d-1)}/2^{d-1}$, so

$$\mathbb{E}[(\rho x + M_{d-1})^{2(d-1)}] \;\le\; 4^{d-1}\big(1 + 2^{-(d-1)}\big)(\rho x)^{2(d-1)} \;\le\; 8^{d-1}(\rho x)^{2(d-1)}\,, \tag{A.4b}$$

i.e., $C_2 = 8$ in the notation of (A.4b). Therefore

$$\mathbb{E}[(\rho x + M_{d-1})^{d-1} \mathbb{1}_{\mathcal{E}^c}] \;\le\; (2\sqrt{2})^{d-1}\, (\rho x)^{d-1}\, \sqrt{\delta_{\mathrm{BDG}}(\rho E)}\,.$$

*(4c) Denominator lower bound:* $\mathbb{E}[|\det|] \ge (\rho x)^{d-1}/2^d$. For any deterministic $R \in (0, \rho x)$, the bulk argument gives

$$\mathbb{E}[|\det(G - \rho x I)|] \;\ge\; (\rho x - R)^{d-1}\, \mathbb{P}(M_{d-1} \le R)\,.$$

Choose $R = \rho x/2$. Then $\rho x - R = \rho x/2$. On $\rho x \ge 8\sqrt{2(d-1)}$, $\rho x/2 \ge 4\sqrt{2(d-1)}$, so Lemma 13 applies at $R = \rho x/2$:

$$\mathbb{P}(M_{d-1} \le \rho x/2) \;\ge\; 1 - e^{-2(\rho x/2)^2/9} \;=\; 1 - e^{-(\rho x)^2/18} \;\ge\; 1 - e^{-128(d-1)/18} \;\ge\; 1 - e^{-64/9} \;\ge\; 1/2\,,$$

for every $d \ge 2$, since $e^{-64/9} \approx 8 \times 10^{-4} < 1/2$. Hence

$$\mathbb{E}[|\det(G_{d-1} - \rho x I_{d-1})|] \;\ge\; \frac{1}{2}\,(\rho x/2)^{d-1} \;=\; (\rho x)^{d-1}/2^d \qquad \text{on } \rho x \ge 8\sqrt{2(d-1)}. \tag{A.4c}$$

*(4d) Combining:* $C_{\mathrm{amp}} = 8\sqrt{2}$. Dividing (4b) by (4c),

$$\frac{\mathbb{E}\big[|\det(G - \rho x I)| \mathbb{1}_{\mathcal{E}^c}\big]}{\mathbb{E}\big[|\det(G - \rho x I)|\big]} \;\le\; \frac{(2\sqrt{2})^{d-1}(\rho x)^{d-1}\sqrt{\delta_{\mathrm{BDG}}(\rho E)}}{(\rho x)^{d-1}/2^d} \;=\; 2 \cdot (4\sqrt{2})^{d-1}\, \sqrt{\delta_{\mathrm{BDG}}(\rho E)}\,.$$

The leading factor $2 \cdot (4\sqrt{2})^{d-1}$ is absorbed into $(8\sqrt{2})^{d-1} = 2^{d-1}(4\sqrt{2})^{d-1} \geq 2 \cdot (4\sqrt{2})^{d-1}$ for $d \geq 2$, giving the explicit bound

$$\frac{\mathbb{E}\big[|\det(G - \rho x I)| \mathbb{1}_{\mathcal{E}^c}\big]}{\mathbb{E}\big[|\det(G - \rho x I)|\big]} \;\leq\; C_{\text{amp}}^{d-1} \sqrt{\delta_{\text{BDG}}(\rho E)}\,, \qquad C_{\text{amp}} := 8\sqrt{2}\,. \tag{A.4d}$$

Combining with $\mathbb{E}[|\det| \mathbb{1}\{G - \rho x I \prec 0\}] \geq \mathbb{E}[|\det|](1 - C_{\text{amp}}^{d-1}\delta_{\text{BDG}}^{1/2}(\rho E))$ established above and integrating over $x \in [E, \infty)$ against $C_{k,d}\varphi(x)$, then applying the identity $C_{k,d}\int_E^\infty \mathbb{E}[|\det(G - \rho x I)|]\varphi(x)\mathrm{d}x = \delta_{\text{exact}}(E)$ from Step 1,

$$\mathbb{E}[N^{\text{lm}}_{[E,\infty)}] \;\geq\; \big(1 - C_{\text{amp}}^{d-1}\delta_{\text{BDG}}^{1/2}(\rho E)\big)\, \delta_{\text{exact}}(E)\,,$$

which is the lower bound in (4.1c). The amplifier $C_{\text{amp}}^{d-1}\delta_{\text{BDG}}^{1/2}(\rho E)$ is super-exponentially small on the validity range: $\sqrt{\delta_{\text{BDG}}(\rho E)} = e^{-(\rho E)^2/9}$ and $(\rho E)^2 \geq 128(d-1)$ give $\delta_{\text{BDG}}^{1/2} \leq e^{-128(d-1)/9}$, while $C_{\text{amp}} = 8\sqrt{2}$ gives $\log C_{\text{amp}} \approx 2.42$; hence $C_{\text{amp}}^{d-1}\delta_{\text{BDG}}^{1/2} \leq [C_{\text{amp}} e^{-128/9}]^{d-1} \leq [8\sqrt{2} \cdot 6.66 \times 10^{-7}]^{d-1} \leq [7.6 \times 10^{-6}]^{d-1}$.

• *Step 3: High-dimensional limit via the ABC complexity theorem.* We avoid redoing the asymptotic of $\delta_{\text{exact}}$ from scratch via the Mehta–Fyodorov decomposition, which would require fine control of the partial Hermite sum $\Phi_d(\rho, E)$ at the natural scale $\rho E = e\sqrt{2(d-1)} \asymp \sqrt{d}$, where the leading-monomial approximation $\Phi_d \sim 2\rho(2\rho E)^{d-2}$ is invalid (every term in the sum contributes at comparable order). Instead, we reduce (4.4b) directly to Auffinger et al. (2013, Theorem 2.4) via an explicit field/level correspondence between our paper's KSS field $X$ on $\mathbb{S}^{d-1}$ and ABC's spherical $k$-spin Hamiltonian $H_{N,p}$ on $S^{N-1}(\sqrt{N})$. The argument has four steps.

*(3a) Field and Hessian correspondence.* Auffinger et al. (2013) work, throughout their Section 3, with the rescaled field

$$f_{N,p}(\boldsymbol{\theta}) \;:=\; \frac{1}{\sqrt{N}}\, H_{N,p}(\sqrt{N}\,\boldsymbol{\theta})\,, \qquad \boldsymbol{\theta} \in \mathbb{S}^{N-1}\,, \tag{A.5a}$$

of unit variance on the unit sphere, with covariance $\mathbb{E}[f(\boldsymbol{\theta})f(\boldsymbol{\theta}')] = \langle \boldsymbol{\theta}, \boldsymbol{\theta}' \rangle^p$ (Auffinger et al., 2013, eq. (3.1)). Identifying $N = d$, $p = k$, the law of $f$ on $\mathbb{S}^{d-1}$ coincides with that of our paper's $X$. ABC's conditional-Hessian formula (Auffinger et al., 2013, Lemma 3.2, part (b)) is stated directly in this unit-sphere setting: for every $\boldsymbol{\theta} \in \mathbb{S}^{N-1}$ and every $x \in \mathbb{R}$,

$$\nabla^2 f(\boldsymbol{\theta}) \,\big|\, f(\boldsymbol{\theta}) = x \;\stackrel{d}{=}\; M^{N-1}\sqrt{2(N-1)\,p(p-1)} \;-\; x\,p\,I_{N-1}\,, \tag{A.5b}$$

where $M^{N-1}$ is a GOE in the ABC normalisation $\mathbb{E}[(M^{N-1}_{ij})^2] = (1 + \delta_{ij})/(2(N-1))$. Setting $G_{N-1} := \sqrt{N-1}\,M^{N-1}$, which has the Mehta normalisation $\mathbb{E}[(G_{N-1})^2_{ij}] = (1 + \delta_{ij})/2$, and using $p/\sqrt{2p(p-1)} = \sqrt{p/(2(p-1))} =: \rho$, the right-hand side rearranges to $\sqrt{2p(p-1)}\,(G_{N-1} - \rho\, x\, I_{N-1})$. With $(N, p) = (d, k)$ this is exactly the conditional Hessian law (2.1a) used in Step 1

$$\nabla^2 X(\boldsymbol{\theta}) \,\big|\, X(\boldsymbol{\theta}) = x \;\stackrel{d}{=}\; \sqrt{2k(k-1)}\,(G_{d-1} - \rho\, x\, I_{d-1})\,, \qquad \rho = \sqrt{k/(2(k-1))}\,,$$

recovering the conditional shifted-GOE law used throughout Section 2. No metric rescaling of the Hessian under a sphere-radius change is needed: ABC perform the rescaling once in passing from $H_{N,p}$ on $S^{N-1}(\sqrt{N})$ to $f_{N,p}$ on $\mathbb{S}^{N-1}$ via (A.5a), and their Hessian formula (A.5b) is already on the unit sphere.

*(3b) Level correspondence.* The threshold $X(\theta) \geq E$ in our paper corresponds to $H(\sigma) \geq E\sqrt{N}$, i.e., $H/N \geq E/\sqrt{N} =: u_{\text{ABC}}$. On the scale $E = e\sqrt{2(d-1)}/\rho$, using $\rho E_\infty = \sqrt{2}$ (which follows from $\rho^2 = k/(2(k-1))$ and $E_\infty^2 = 4(k-1)/k$),

$$u_{\text{ABC}} \;=\; \frac{E}{\sqrt{d}} \;=\; \frac{e\sqrt{2(d-1)}}{\rho\sqrt{d}} \;\xrightarrow[d\to\infty]{}\; \frac{e\sqrt{2}}{\rho} \;=\; \frac{e\sqrt{2}}{\rho E_\infty} \cdot E_\infty \;=\; e\,E_\infty\,. \tag{A.5c}$$

For $e \geq 1$, $u_{\text{ABC}} \to eE_\infty \geq E_\infty$, so $-u_{\text{ABC}} \leq -E_\infty$ and the case-1 branch of (4.4a) applies.

*(3c) Asymptotic rate for the total critical-point count.* By Auffinger et al. (2013, Theorem 2.4) (log-rate for the total critical-point count below level $Nu$) combined with the upper-tail symmetry $H \stackrel{d}{=} -H$ (an immediate consequence of $J \stackrel{d}{=} -J$ in the disorder), for every fixed $u > 0$:

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\big[\#\{\sigma : \nabla H(\sigma) = 0,\; H(\sigma) \in (Nu, \infty)\}\big] \;=\; \Theta_k(-u)\,.$$

By the Kac–Rice equality of Proposition 1 (recall from Step 1 that $\delta_{\text{exact}}(E)$ equals the Kac–Rice expected critical-point count above level $E$), the left-hand side equals $\lim_d \frac{1}{d} \log \delta_{\text{exact}}(E)$ under the field/level correspondence of (3a)–(3b). Hence

$$\lim_{d\to\infty} \frac{1}{d} \log \delta_{\text{exact}}(E) \;=\; \Theta_k(-eE_\infty) \;=\; \tfrac{1}{2}\log(k-1) - \tfrac{(k-2)e^2}{k} - 2\int_1^e \sqrt{w^2-1}\,\mathrm{d}w \tag{A.5d}$$

for $e \geq 1$, using (4.4c).

*(3d) Local-maxima specialisation and the bracketing.* ABC's Theorem 2.4 covers the total critical-point count; for local maxima (Morse index $N-1$ on the sphere) specifically, the symmetry remark following Auffinger et al. (2013, Theorem 2.3) (their Remark 3.1) gives $\lim_N \frac{1}{N} \log \mathbb{E}[\#\{\text{local maxima of } H \text{ above } Nu\}] = \Theta_{0,p}(-u)$, and the case-1 branch $\Theta_{0,p}(u) = \Theta_p(u)$ on $u \leq -E_\infty$ (Auffinger et al., 2013, Eq. (2.14) and Eq. (2.16)) states that for $e \geq 1$, the local-maxima rate coincides with the total-critical-point rate: every critical point above the spectral edge is, with overwhelming probability, a local maximum (the conditional Hessian $G_{d-1} - \rho x I$ is strictly negative-definite on the high-probability event $M_{d-1} < \rho x$, which has probability $1 - o(1)$ for $\rho x \geq 2\sqrt{d-1}$). Therefore

$$\lim_{d\to\infty} \frac{1}{d} \log \mathbb{E}[N^{\text{lm}}_{[E,\infty)}] \;=\; \Theta_k(-eE_\infty).$$

Equivalently, and consistently with our non-asymptotic bracketing of (4.1c), combining the upper bound $\mathbb{E}[N^{\text{lm}}] \leq \delta_{\text{exact}}$ with the lower bound $\mathbb{E}[N^{\text{lm}}] \geq (1 - C_{\text{amp}}^{d-1}\delta_{\text{BDG}}^{1/2})\delta_{\text{exact}}$ (valid for $e \geq 8$), and observing that on $\rho E \geq 8\sqrt{2(d-1)}$ the amplifier $C_{\text{amp}}^{d-1}\delta_{\text{BDG}}^{1/2}(\rho E) \leq [C_{\text{amp}} e^{-128/9}]^{d-1}$ vanishes super-exponentially as $d \to \infty$,

$$\tfrac{1}{d}\log \delta_{\text{exact}} + \tfrac{1}{d}\log\left(1 - C_{\text{amp}}^{d-1}\delta_{\text{BDG}}^{1/2}\right) \;\leq\; \tfrac{1}{d}\log \mathbb{E}[N^{\text{lm}}] \;\leq\; \tfrac{1}{d}\log \delta_{\text{exact}}\,,$$

both bracketing terms converge to $\Theta_k(-eE_\infty)$ by (A.5d). The limit (4.4b) is thus established for every $e \geq 1$ by the exact Kac–Rice equality of Step 1 and Auffinger et al. (2013, Theorem 2.4) (Steps 3c–3d); on the smaller range $e \geq 8$ the non-asymptotic two-sided bracket above re-derives the same limit with explicit finite-$d$ error bars. The moderate-energy regime $1 \leq e < 8$ is therefore covered by the asymptotic statement (4.4b), while only the matching finite-$d$ *lower* bracket – not the limit itself – lies outside the scope of the present argument. ☐

# B Inversion of the SMF tail bound: proof of Remark 1

This appendix gives the full derivation of the level $u_\alpha$ of (1.8) that ensures $\delta_{\min}(u_\alpha) \leq \alpha$. We isolate the SMF asymptotic single-term form (Corollary 3), invert it term-by-term, and identify the explicit threshold $\alpha_0(k,d)$ above which the closed-form choice (1.8) is sufficient.

Throughout the appendix we set $n := d-1$ to lighten notation and write $\ell := \log(1/\alpha) > 0$, so that (1.8) reads

$$u_\alpha^2 \;=\; 2\ell + n\log(2k) + 2n\log\ell\,. \tag{B.1}$$

**Proposition 10** (Explicit inversion of $\delta_{\min}$)**.** *Let $k \geq 3$, $d \geq 3$, and let $\alpha_d, \beta_d, u_d^\star$ be the explicit constants of Lemma 5 and Corollary 3. Define*

$$\alpha_0(k,d) \;:=\; \min\Big\{\tfrac{1}{2}\,e^{-(u_d^\star)^2/2}\,,\; e^{-n}\,,\; \exp\big(-\tfrac{n\log(2k)}{2}\big)\,,\; \exp\big(-16\,\alpha_d\,(2e)^{(n-1)/2}\big)\Big\}. \tag{B.2}$$

*Then for every $\alpha \in (0, \alpha_0(k,d)]$, the level $u_\alpha$ defined in (B.1) satisfies*

$$\delta_{\min}(u_\alpha) \;\leq\; \alpha\,.$$

*Proof.* • For $u \geq u_d^\star \geq u_{\text{SMF}} = 2\sqrt{d}$, Corollary 3 gives the one-sided single-term bound

$$\mathbb{P}\Big\{\sup_{\boldsymbol{\theta}\in\mathbb{S}^{d-1}} X(\boldsymbol{\theta}) > u\Big\} \;\leq\; 8\,\alpha_d\,(2k)^{n/2}\,u^{n-1}\,e^{-u^2/2}\,. \tag{B.3a}$$

By definition $\delta_{\min}(u) \le 2\,\delta_{\mathsf{SMF}}(u)$ for $u \ge u_{\mathsf{SMF}}$, and Corollary 3 yields $\delta_{\mathsf{SMF}}(u) \le 8\alpha_d(2k)^{n/2}u^{n-1}e^{-u^2/2}$ on $[u_d^\star, \infty)$. Multiplying by the symmetry factor 2,

$$u \ge u_d^\star \implies \delta_{\min}(u) \;\le\; C^\star_{k,d}\, u^{n-1}\, e^{-u^2/2}\,, \qquad C^\star_{k,d} := 16\,\alpha_d\,(2k)^{n/2}\,. \tag{B.3b}$$

Hence, to prove the proposition it suffices to find $u_\alpha \ge u_d^\star$ with $C^\star_{k,d}\, u_\alpha^{n-1}\, e^{-u_\alpha^2/2} \le \alpha$, i.e.

$$\frac{u_\alpha^2}{2} - (n-1)\log u_\alpha \;\ge\; \log(1/\alpha) + \log C^\star_{k,d} \;=\; \ell + \log(16\,\alpha_d) + \tfrac{n}{2}\log(2k)\,. \tag{B.3c}$$

• Set $s := u_\alpha^2$ and $g(s) := s/2 - (n-1)\log\sqrt{s} = s/2 - \frac{n-1}{2}\log s$. Inequality (B.3c) is exactly $g(s) \ge \ell + \log(16\,\alpha_d) + \frac{n}{2}\log(2k)$. The function $g$ is increasing on $[n-1, \infty)$ and grows like $s/2$ for large $s$, while the subtracted term $\frac{n-1}{2}\log s$ is logarithmically slow. We will choose $s$ large enough that this logarithmic correction is dominated by the slack term $2n\log\ell$ in the ansatz (B.1).

• Plug $s = 2\ell + n\log(2k) + 2n\log\ell$ from (B.1) into $g(s)$:

$$g(s) \;=\; \ell + \tfrac{n}{2}\log(2k) + n\log\ell - \tfrac{n-1}{2}\log\bigl(2\ell + n\log(2k) + 2n\log\ell\bigr)\,. \tag{B.4a}$$

The first three terms exactly produce the right-hand side of (B.3c) up to two corrections: a positive slack of $n\log\ell$ on the left, and a missing summand $\log(16\,\alpha_d)$ on the right. We must absorb the remaining $\log(2\ell+n\log(2k)+2n\log\ell)$ on the left. Using $\log(a+b+c) \le \log a + \log(1+b/a+c/a) \le \log a + (b+c)/a$ for $a > 0$ and $b, c \ge 0$, with $a = 2\ell$:

$$\log\bigl(2\ell + n\log(2k) + 2n\log\ell\bigr) \;\le\; \log(2\ell) + \frac{n\log(2k) + 2n\log\ell}{2\ell}\,. \tag{B.4b}$$

Substituting (B.4b) into (B.4a),

$$g(s) \;\ge\; \ell + \tfrac{n}{2}\log(2k) + n\log\ell - \tfrac{n-1}{2}\log(2\ell) - \tfrac{n-1}{2}\cdot\frac{n\log(2k) + 2n\log\ell}{2\ell}\,. \tag{B.4c}$$

• Subtracting the right-hand side of (B.3c) from (B.4c), the residual reads

$$\mathcal{R}(\alpha; k, d) \;:=\; n\log\ell - \log(16\,\alpha_d) - \tfrac{n-1}{2}\log(2\ell) - \tfrac{n-1}{4\ell}\bigl(n\log(2k) + 2n\log\ell\bigr)\,. \tag{B.5}$$

The proof is complete once we show that $\mathcal{R}(\alpha; k, d) \ge 0$ for $\alpha \in (0, \alpha_0(k, d)]$. Each clause of (B.2) translates into a lower bound on $\ell = \log(1/\alpha)$, and we combine them to bound the two negative "correction" contributions in (B.5) explicitly.

*(a)* The clause $\alpha \le e^{-n}$ gives $\ell \ge n$, hence $n/\ell \le 1$ and the $\log\ell$-correction obeys

$$\tfrac{n-1}{4\ell}\cdot 2n\log\ell \;=\; \tfrac{n(n-1)\log\ell}{2\ell} \;\le\; \tfrac{n-1}{2}\log\ell\,.$$

*(b)* The clause $\alpha \le e^{-n\log(2k)/2}$ gives $2\ell \ge n\log(2k)$, hence the $\log(2k)$-correction obeys $\frac{n-1}{4\ell}\, n\log(2k) \le \frac{n-1}{4\ell}\cdot 2\ell = \frac{n-1}{2}$.

Substituting (a) and (b) into (B.5) and using $\log(2\ell) = \log 2 + \log\ell$,

$$\begin{aligned}\mathcal{R}(\alpha; k, d) \;&\ge\; n\log\ell - \log(16\,\alpha_d) - \tfrac{n-1}{2}\log\ell - \tfrac{n-1}{2}\log 2 - \tfrac{n-1}{2}\log\ell - \tfrac{n-1}{2}\\ &=\; \log\ell - \log(16\,\alpha_d) - \tfrac{n-1}{2}(1 + \log 2)\,,\end{aligned}$$

since $n\log\ell - (n-1)\log\ell = \log\ell$. The clause $\alpha \le \exp(-16\,\alpha_d(2e)^{(n-1)/2})$ gives $\ell \ge 16\,\alpha_d(2e)^{(n-1)/2}$, hence $\log\ell \ge \log(16\,\alpha_d) + \frac{n-1}{2}(1+\log 2)$ (as $\log(2e) = 1 + \log 2$), so the bound above is $\ge 0$. Finally, the clause $\alpha \le \frac{1}{2}e^{-(u_d^\star)^2/2}$ guarantees $u_\alpha \ge u_d^\star$ (the prerequisite for the single-term reduction (B.3b)) since $u_\alpha^2 \ge 2\ell \ge (u_d^\star)^2$ on this range. Combining the four clauses, $\mathcal{R}(\alpha; k, d) \ge 0$ on $(0, \alpha_0(k, d)]$. □

**Remark 10** (Explicitness of the threshold $\alpha_0(k, d)$)**.** *The threshold $\alpha_0(k, d)$ defined in (B.2) is fully explicit: it depends on $(k, d)$ only through the constants $\alpha_d$, $\beta_d$, $u_d^\star$ of Lemma 5 and Corollary 3, with no further hidden quantities. In particular, $\alpha_0(k, d) > 0$ for every fixed $(k, d)$, so (1.8) is a sufficient choice on the entire confidence range $(0, \alpha_0(k, d)]$. Larger levels, including $\alpha = 0.05$ (which generally exceeds $\alpha_0(k, d)$, since $\alpha_0(k, d) \le \frac{1}{2}e^{-(u_d^\star)^2/2} \le \frac{1}{2}e^{-2d}$ as $u_d^\star \ge 2\sqrt{d}$), are handled by the numerical inversion of Remark 4.*

# C Asymptotic Analysis of the Tail Bound Prefactors

In this section, we analyze the prefactors of the three tail bounds presented in Table 1. We first isolate the exact prefactors for fixed $d$ and $k$ in the large-$u$ regime ($u \to \infty$), and then evaluate their high-dimensional asymptotic behavior ($d \to \infty$) using Stirling's approximation to compare them against the exact Kac–Rice baseline.

## C.1 Fixed $(k, d)$ prefactors as $u \to \infty$

As $u \to \infty$, the tail bounds and the baseline all decay at the leading rate of $u^{d-2}e^{-u^2/2}$. We define the prefactor $P$ for each method such that the tail probability behaves as $P \cdot u^{d-2}e^{-u^2/2}$.

**Asymptotic Baseline.** From the exact limit of the Kac–Rice integral, the optimal baseline prefactor is:

$$P_{\text{base}} = \frac{\sqrt{2}}{\Gamma(d/2)} \left(\frac{k}{2}\right)^{\frac{d-1}{2}}. \tag{C.1a}$$

**Spectral Method (SM).** The SM tail bound is $2\frac{\sqrt{2}}{\Gamma(d/2)}\left(\frac{k}{2}\right)^{\frac{d-1}{2}} u^{d-2}e^{-u^2/2}(1+\eta_d(\rho,u))$. Since the correction term $\eta_d(\rho, u) \to 0$ as $u \to \infty$, the SM prefactor is twice the baseline:

$$P_{\text{SM}} = 2\frac{\sqrt{2}}{\Gamma(d/2)} \left(\frac{k}{2}\right)^{\frac{d-1}{2}} = 2P_{\text{base}}. \tag{C.1b}$$

**Improved Mehta–Fyodorov (IMF) Bound.** The IMF bound is driven by the polynomial-exponential function $\Phi_d(\rho, u)$. For large $u$, the leading monomial of $\Phi_d(\rho, u)$ (corresponding to $k = 0$ in its series) is $2\rho(2\rho u)^{d-2}$. Multiplying this by the global terms in the IMF bound gives the leading behavior:

$$2(k-1)^{\frac{d-1}{2}}\alpha_d \left[2\rho(2\rho u)^{d-2}\right] e^{-u^2/2} = 4\alpha_d \rho^{d-1}(k-1)^{\frac{d-1}{2}} 2^{d-2}u^{d-2}e^{-u^2/2}.$$

Using the definition of the shift parameter $\rho = \sqrt{k/(2(k-1))}$, we simplify the algebraic constants:

$$\rho^{d-1}(k-1)^{\frac{d-1}{2}} = \left(\frac{k}{2(k-1)}\right)^{\frac{d-1}{2}} (k-1)^{\frac{d-1}{2}} = \left(\frac{k}{2}\right)^{\frac{d-1}{2}}.$$

Substituting this back yields the IMF prefactor:

$$P_{\text{IMF}} = 4\alpha_d \left(\frac{k}{2}\right)^{\frac{d-1}{2}} 2^{d-2} = 2\alpha_d (2k)^{\frac{d-1}{2}}. \tag{C.1c}$$

**Simplified Mehta–Fyodorov (SMF) Bound.** By Corollary 3, the remainder term $e^{-3u^2/4}$ of the SMF bound vanishes for large $u$, leaving the single main term $4\alpha_d(2k)^{\frac{d-1}{2}} u^{d-2}e^{-u^2/2}$. Thus, its prefactor is:

$$P_{\text{SMF}} = 4\alpha_d(2k)^{\frac{d-1}{2}} = 2P_{\text{IMF}}. \tag{C.1d}$$

The factor of 2 penalty compared to the IMF bound arises directly from replacing the partial Hermite sum with the looser Szegő envelope $H_{d-1}(x) < (2x)^{d-1}$.

## C.2 High-dimensional asymptotics as $d \to \infty$

To compare the methods in high dimensions, we expand $P_{\text{base}}$ and $P_{\text{IMF}}$ as $d \to \infty$ (keeping $k$ fixed) using Stirling's approximation.

**Expanding the Baseline.** Using Stirling's approximation for the Gamma function, $\Gamma(z) \sim \sqrt{2\pi} z^{z-1/2} e^{-z}$, we expand $\Gamma(d/2)$:

$$\Gamma(d/2) \sim \sqrt{2\pi} \left(\frac{d}{2}\right)^{\frac{d-1}{2}} e^{-d/2} = \sqrt{\frac{4\pi}{d}} \left(\frac{d}{2e}\right)^{d/2}.$$

Substituting this into $P_{\text{base}}$ yields:

$$P_{\text{base}} \sim \frac{\sqrt{2}}{\sqrt{4\pi/d}\,\left(\frac{d}{2e}\right)^{d/2}} \frac{(k/2)^{d/2}}{\sqrt{k/2}} = \sqrt{\frac{d}{\pi k}} \left(\frac{ek}{d}\right)^{d/2}. \tag{C.2a}$$

**Expanding the Mehta–Fyodorov Prefactor.** From Lemma 5, the Stirling approximation for the dominant Hermite coefficient $\alpha_d$ (taking $d$ even for simplicity) is:

$$\alpha_d \sim \sqrt{\frac{d}{2\pi}} \left(\frac{e}{2d}\right)^{d/2}.$$

Substituting this into $P_{\text{IMF}}$ yields:

$$P_{\text{IMF}} = 2\alpha_d (2k)^{\frac{d-1}{2}} \sim 2\sqrt{\frac{d}{2\pi}} \left(\frac{e}{2d}\right)^{d/2} \frac{(2k)^{d/2}}{\sqrt{2k}} = \sqrt{\frac{d}{\pi k}} \left(\frac{ek}{d}\right)^{d/2}. \tag{C.2b}$$

**Conclusion.** Comparing (C.2a) and (C.2b), as $d \to \infty$ one has $P_{\text{IMF}} \sim P_{\text{base}}$. The Improved Mehta–Fyodorov bound captures the exact leading Stirling behavior and constant factor of the Kac–Rice integral without any loss. In contrast, both $P_{\text{SM}}$ and $P_{\text{SMF}}$ satisfy:

$$P_{\text{SM}} \sim P_{\text{SMF}} \sim 2\sqrt{\frac{d}{\pi k}} \left(\frac{ek}{d}\right)^{d/2}. \tag{C.2c}$$

All bounds capture the exponential scale $\left(\frac{ek}{d}\right)^{d/2}$; the SM and SMF bounds pay a uniform factor-2 penalty compared to the optimal baseline, while the IMF bound is sharp.

# D Technical lemmas

## D.1 Gaussian-type tail integrals

The proofs of the three tail bounds use a common tool: integration by parts on tail integrals of the form $\int_u^\infty x^m e^{-ax^2/2}\,\mathrm{d}x$, iterated in steps of two. We isolate it here.

**Lemma 10** (Gaussian-type tail integrals)**.** *For $a > 0$, integer $m \geq 0$, and $u > 0$, define $I_m(u; a) := \int_u^\infty x^m e^{-ax^2/2}\,\mathrm{d}x$. The base case $m = 0$ obeys the Mills-type bound*

$$I_0(u; a) \;=\; \int_u^\infty e^{-ax^2/2}\,\mathrm{d}x \;\leq\; \frac{e^{-au^2/2}}{a\,u}, \tag{D.1a}$$

*and $I_1(u; a) = a^{-1} e^{-au^2/2}$ exactly. For every integer $m \geq 2$,*

$$I_m(u; a) \;=\; \frac{u^{m-1}\,e^{-au^2/2}}{a} \;+\; \frac{m-1}{a}\, I_{m-2}(u; a), \tag{D.1b}$$

*and, provided $(m-1)/(au^2) < 1$,*

$$I_m(u; a) \;\leq\; \frac{u^{m-1}\,e^{-au^2/2}}{a - (m-1)/u^2}. \tag{D.1c}$$

*In particular,*

$$\begin{aligned} I_{d-1}(u; 1) &\leq 2u^{d-2} e^{-u^2/2}, && \text{for } u^2 \geq 2(d-2), \\ I_{2d-2}(u; 3/2) &\leq u^{2d-3} e^{-3u^2/4}, && \text{for } u \geq 2\sqrt{d}, \\ I_{d-2}(u; 4/9) &\leq 3\,u^{d-3} e^{-2u^2/9} \leq u^{d-1} e^{-2u^2/9}, && \text{for } u \geq 4\sqrt{2(d-1)}. \end{aligned}$$

*Proof. Base case $m = 0$.* On $[u, \infty)$ one has $1 \le x/u$, hence

$$I_0(u; a) = \int_u^\infty e^{-ax^2/2}\,\mathrm{d}x \;\le\; \frac{1}{u}\int_u^\infty x\,e^{-ax^2/2}\,\mathrm{d}x \;=\; \frac{1}{u}\cdot\frac{e^{-au^2/2}}{a} \;=\; \frac{e^{-au^2/2}}{a\,u},$$

which is (D.1a); the case $m = 1$ is the displayed exact evaluation. For $m \ge 2$ we want to evaluate the integral

$$I_m(u; a) = \int_u^\infty x^m e^{-ax^2/2}\,\mathrm{d}x.$$

We integrate by parts, splitting the integrand as $x^{m-1} \cdot xe^{-ax^2/2}$ and choosing:

$$f(x) = x^{m-1} \implies f'(x) = (m-1)x^{m-2}$$

$$g'(x) = xe^{-ax^2/2} \implies g(x) = -\frac{1}{a}e^{-ax^2/2}$$

Applying the integration by parts formula $\int f\,\mathrm{d}g = fg - \int g\,\mathrm{d}f$:

$$I_m(u; a) = \left[-x^{m-1}\frac{1}{a}e^{-ax^2/2}\right]_u^\infty - \int_u^\infty \left(-\frac{1}{a}e^{-ax^2/2}\right)(m-1)x^{m-2}\,\mathrm{d}x.$$

Since $a > 0$, the exponential term dominates the polynomial term as $x \to \infty$, making the upper boundary evaluate to 0. Evaluating at the lower bound $u$ gives:

$$I_m(u; a) = \frac{u^{m-1}e^{-au^2/2}}{a} + \frac{m-1}{a}\int_u^\infty x^{m-2}e^{-ax^2/2}\,\mathrm{d}x.$$

The remaining integral is $I_{m-2}(u; a)$, giving the recurrence:

$$I_m(u; a) = \frac{u^{m-1}\,e^{-au^2/2}}{a} + \frac{m-1}{a}\,I_{m-2}(u; a).$$

We iterate this recurrence relation to express $I_m(u; a)$ as a series:

$$I_m(u; a) = \frac{e^{-au^2/2}}{a}\left[u^{m-1} + \frac{m-1}{a}u^{m-3} + \frac{(m-1)(m-3)}{a^2}u^{m-5} + \dots\right].$$

Factoring out $u^{m-1}$, we get:

$$I_m(u; a) = \frac{u^{m-1}e^{-au^2/2}}{a}\left[1 + \frac{m-1}{au^2} + \frac{(m-1)(m-3)}{(au^2)^2} + \dots\right].$$

The numerator coefficients satisfy $m-3 < m-1$, $m-5 < m-1$, and so on. Replacing these descending factors by $m-1$ gives a strict upper bound:

$$\frac{(m-1)(m-3)}{(au^2)^2} \le \left(\frac{m-1}{au^2}\right)^2.$$

Applying this bound to all terms allows us to upper bound the finite sum by an infinite geometric series:

$$I_m(u; a) \le \frac{u^{m-1}e^{-au^2/2}}{a}\sum_{k=0}^\infty \left(\frac{m-1}{au^2}\right)^k.$$

By hypothesis $\frac{m-1}{au^2} < 1$, so the geometric series converges, yielding:

$$I_m(u; a) \le \frac{u^{m-1}e^{-au^2/2}}{a}\left(\frac{1}{1 - \frac{m-1}{au^2}}\right).$$

which gives the closed-form bound:

$$I_m(u; a) \le \frac{u^{m-1}e^{-au^2/2}}{a - (m-1)/u^2}.$$

□

## D.2 Asymptotic leading term

**Lemma 11** (Asymptotic Equivalence of the Expected Determinant)**.** *Let $G_{d-1} \sim GOE(d-1)$ with eigenvalues $\mu_1, \dots, \mu_{d-1}$. As $x \to \infty$, the expected absolute characteristic polynomial is asymptotically equivalent to its leading monomial:*

$$\mathbb{E}\Big[\prod_{i=1}^{d-1} |\mu_i - \rho x|\Big] \sim (\rho x)^{d-1}. \tag{D.2}$$

*Proof.* Let $M_{d-1} = \max_i |\mu_i|$ be the spectral radius of the GOE matrix. We split the expectation using a threshold $R(x) = \sqrt{\rho x \sqrt{d-1}}$.

• We partition the expectation into a bulk event $\{M_{d-1} \le R\}$ and a tail event $\{M_{d-1} > R\}$. On the bulk event, by the triangle inequality, $|\mu_i - \rho x| \le R + \rho x$. On the tail event, we apply the Ben Arous–Dembo–Guionnet large deviation bound of Lemma 13, which guarantees $\mathbb{P}(M_{d-1} > t) \le e^{-2t^2/9}$ for every $t \ge 4\sqrt{2(d-1)}$. As established in Proposition 4, balancing these two regimes yields:

$$\mathbb{E}\Big[\prod_{i=1}^{d-1} |\mu_i - \rho x|\Big] \le (\rho x)^{d-1}\big(1 + \eta_d(\rho x)\big), \tag{D.3a}$$

where $\eta_d(\rho x) = \mathcal{O}\left(x^{-1/2}\right)$ as $x \to \infty$, meaning $\eta_d(\rho x) \to 0$.

• Since the integrand is non-negative, we lower-bound it by dropping the tail event and integrating over the bulk:

$$\mathbb{E}\Big[\prod_{i=1}^{d-1} |\mu_i - \rho x|\Big] \ge \mathbb{E}\Big[\prod_{i=1}^{d-1} |\mu_i - \rho x| \mathbb{1}_{\{M_{d-1} \le R\}}\Big]. \tag{D.3b}$$

On $\{M_{d-1} \le R\}$, $|\mu_i| \le R$ for every $i$, so $\rho x - \mu_i \ge \rho x - R$. For $x$ large enough that $\rho x > R$ (which holds since $R(x) = \sqrt{\rho x \sqrt{d-1}} = o(\rho x)$), $\rho x - \mu_i > 0$, so the absolute value may be dropped: $|\mu_i - \rho x| = \rho x - \mu_i \ge \rho x - R$. Therefore:

$$\mathbb{E}\Big[\prod_{i=1}^{d-1} |\mu_i - \rho x|\Big] \ge (\rho x - R)^{d-1} \mathbb{P}(M_{d-1} \le R). \tag{D.3c}$$

Factoring out $(\rho x)^{d-1}$, we get:

$$\mathbb{E}\Big[\prod_{i=1}^{d-1} |\mu_i - \rho x|\Big] \ge (\rho x)^{d-1} \left(1 - \frac{R}{\rho x}\right)^{d-1} \big(1 - \mathbb{P}(M_{d-1} > R)\big). \tag{D.3d}$$

Since $R = \sqrt{\rho x \sqrt{d-1}}$, the ratio $\frac{R}{\rho x} \to 0$ as $x \to \infty$. Furthermore, by the super-exponential tail bound, $\mathbb{P}(M_{d-1} > R) \to 0$.

• Let $E(x) = \mathbb{E}\big[\prod_{i=1}^{d-1} |\mu_i - \rho x|\big]$. We have shown that for large $x$:

$$(\rho x)^{d-1}(1 - o(1)) \le E(x) \le (\rho x)^{d-1}(1 + o(1)). \tag{D.3e}$$

Hence $\lim_{x\to\infty} E(x)/(\rho x)^{d-1} = 1$, i.e. $E(x) \sim (\rho x)^{d-1}$, the asymptotic equivalence used to define the baseline. □

**Lemma 12** (Asymptotic Equivalence of the Tail Integrals)**.** *Let $E(x) = \mathbb{E}\big[|\det(G_{d-1} - \rho x\, I_{d-1})|\big]$. As the threshold $u \to \infty$, the Kac-Rice tail integral is asymptotically equivalent to the baseline integral:*

$$C_{k,d} \int_u^\infty E(x)\, \varphi(x)\, \mathrm{d}x \sim C_{k,d} \int_u^\infty (\rho x)^{d-1}\, \varphi(x)\, \mathrm{d}x. \tag{D.4}$$

*Proof.* To prove the asymptotic equivalence, we must show that the limit of the ratio of the two integrals evaluates to 1 as $u \to \infty$:

$$L = \lim_{u\to\infty} \frac{\int_u^\infty E(x)\, \varphi(x)\, \mathrm{d}x}{\int_u^\infty (\rho x)^{d-1}\, \varphi(x)\, \mathrm{d}x}. \tag{D.5a}$$

The constant $C_{k,d}$ cancels from the ratio, and both integrands are integrable, so numerator and denominator tend to 0 as $u \to \infty$. By L'Hôpital's rule, using $\frac{\mathrm{d}}{\mathrm{d}u}\int_u^\infty f(x)\,\mathrm{d}x = -f(u)$, differentiating numerator and denominator yields:

$$L = \lim_{u\to\infty} \frac{\frac{\mathrm{d}}{\mathrm{d}u}\int_u^\infty E(x)\,\varphi(x)\,\mathrm{d}x}{\frac{\mathrm{d}}{\mathrm{d}u}\int_u^\infty (\rho x)^{d-1}\,\varphi(x)\,\mathrm{d}x} = \lim_{u\to\infty} \frac{-E(u)\,\varphi(u)}{-(\rho u)^{d-1}\,\varphi(u)}. \tag{D.5b}$$

The negative signs cancel, and because the standard normal density $\varphi(u)$ is strictly positive for all $u$, it also cancels out of the ratio. This leaves:

$$L = \lim_{u\to\infty} \frac{E(u)}{(\rho u)^{d-1}}. \tag{D.5c}$$

By Lemma 11, we know that $E(u) \sim (\rho u)^{d-1}$ as $u \to \infty$. By definition of asymptotic equivalence, this means:

$$\lim_{u\to\infty} \frac{E(u)}{(\rho u)^{d-1}} = 1. \tag{D.5d}$$

Since the limit of the ratio of derivatives is 1, so is the limit of the original ratio, giving the asymptotic equivalence:

$$\int_u^\infty E(x)\,\varphi(x)\,\mathrm{d}x \sim \int_u^\infty (\rho x)^{d-1}\,\varphi(x)\,\mathrm{d}x. \tag{D.5e}$$

Multiplying both sides by the global constant $C_{k,d}$ concludes the proof. □

## D.3 The Ben Arous–Dembo–Guionnet lemma

We first record the Ben Arous-Dembo–Guionnet large-deviation bound (Ben Arous et al., 2001, Lemma-6.3) for the spectral radius of the GOE in the Mehta normalization adopted here.

**Lemma 13** (Spectral radius tail). *Let $G_{d-1} \sim \mathrm{GOE}(d-1)$ with eigenvalues $\mu_1, \dots, \mu_{d-1}$, and denote $M_{d-1} = \max_i |\mu_i|$. Then*

$$\mathbb{P}\{M_{d-1} > t\} \;\le\; e^{-2t^2/9} \qquad \textit{for all } t \ge 4\sqrt{2(d-1)}\,.$$

*Proof.* • Let $J_N$ be the symmetric random matrix defined in the Ben Arous–Dembo–Guionnet normalization, with joint eigenvalue density $\sigma^N \propto \prod_{i<j}|\lambda_i - \lambda_j| \exp(-\frac{N}{4}\sum_{i=1}^N \lambda_i^2)$. Isolating $\lambda_1 = x$ and splitting the Gaussian weight as $e^{-\frac{N}{4}\sum_{i=2}^N \lambda_i^2} = e^{-\frac{N-1}{4}\sum_{i=2}^N \lambda_i^2} e^{-\frac{1}{4}\sum_{i=2}^N \lambda_i^2}$, the remaining interaction exactly matches the $(N-1)$-dimensional density $\sigma^{N-1}$. Thus,

$$\sigma^N(\lambda_1 \ge M) = \frac{Z_{N-1}}{Z_N}\int_M^\infty e^{-\frac{N}{4}x^2}\,\mathbb{E}_{\sigma^{N-1}}\left[\prod_{i=2}^N |x-\lambda_i| e^{-\lambda_i^2/4}\right]\mathrm{d}x\,. \tag{D.6a}$$

The partition function $Z_N = \int_{\mathbb{R}^N}\prod_{i<j}|\lambda_i - \lambda_j|\, e^{-\frac{N}{4}\sum_i \lambda_i^2}\,\mathrm{d}\lambda$ of this $\beta = 1$ Gaussian ensemble is computed from the Mehta integral (Mehta, 2004, Ch. 17) (the $\gamma = \frac{1}{2}$ case of $\int_{\mathbb{R}^N}\prod_{i<j}|x_i - x_j|^{2\gamma} e^{-\frac{1}{2}\sum x_i^2}\mathrm{d}x = (2\pi)^{N/2}\prod_{j=1}^N \Gamma(1+j\gamma)/\Gamma(1+\gamma)$):

$$M_N \;:=\; \int_{\mathbb{R}^N}\prod_{i<j}|x_i - x_j|\, e^{-\frac{1}{2}\sum_i x_i^2}\,\mathrm{d}x \;=\; (2\pi)^{N/2}\prod_{j=1}^N \frac{\Gamma(1+j/2)}{\Gamma(3/2)}\,. \tag{D.6b}$$

The rescaling $\lambda_i = \sqrt{2/N}\,x_i$ turns the weight $e^{-\frac{N}{4}\sum \lambda_i^2}$ into $e^{-\frac{1}{2}\sum x_i^2}$, and contributes a Jacobian $(2/N)^{N/2}$ together with a Vandermonde factor $(2/N)^{N(N-1)/4}$ (one factor $\sqrt{2/N}$ per pair, $\binom{N}{2} = \frac{N(N-1)}{2}$ pairs), so that

$$Z_N \;=\; (2/N)^{N(N+1)/4}\,(2\pi)^{N/2}\prod_{j=1}^N \frac{\Gamma(1+j/2)}{\Gamma(3/2)}\,. \tag{D.6c}$$

Taking the ratio at sizes $N-1$ and $N$: the Gamma product telescopes to $P_{N-1}/P_N = \Gamma(3/2)/\Gamma(1+N/2)$ (with $P_N := \prod_{j=1}^N \Gamma(1+j/2)/\Gamma(3/2)$), the Gaussian prefactor contributes $(2\pi)^{-1/2}$, and the dimension factors combine through the exponent identity $\frac{N(N-1)}{4} - \frac{N(N+1)}{4} = -\frac{N}{2}$ into $2^{-N/2}\,(1+\frac{1}{N-1})^{N(N-1)/4}\,N^{N/2}$. Using $(2\pi)^{-1/2}\Gamma(3/2) = \frac{1}{2\sqrt{2}}$ and $2^{-N/2}N^{N/2} = (N/2)^{N/2}$, this yields the exact identity

$$\frac{Z_{N-1}}{Z_N} = \left(1+\frac{1}{N-1}\right)^{\frac{N(N-1)}{4}} \frac{(N/2)^{N/2}}{2\sqrt{2}\,\Gamma(1+N/2)} \le e^{N/4}\,\frac{e^{N/2}}{2\sqrt{2\pi N}} \le e^{3N/4}\,. \tag{D.6d}$$

Here the first inequality bounds the two factors separately: $\left(1+\frac{1}{N-1}\right)^{N(N-1)/4} = \exp\left(\frac{N(N-1)}{4}\log(1+\frac{1}{N-1})\right) \le \exp(\frac{N}{4})$ via $\log(1+x) \le x$, while $\frac{(N/2)^{N/2}}{2\sqrt{2}\,\Gamma(1+N/2)} \le \frac{e^{N/2}}{2\sqrt{2\pi N}}$ uses the standard lower bound $\Gamma(1+N/2) \ge (N/2)^{N/2}e^{-N/2}\sqrt{\pi N}$; the last step uses $2\sqrt{2\pi N} \ge 1$. For the integrand, since $\sup_\lambda |\lambda| e^{-\lambda^2/4} = \sqrt{2/e} \le 1$, we have $|x-\lambda_i|e^{-\lambda_i^2/4} \le x+1$. For $x \ge 8$, $x+1 = x(1+1/x) \le xe^{1/8}$, meaning the product is bounded by $x^{N-1}e^{N/8}$.

- Combining the bounds, the probability is bounded by the tail integral:

$$\sigma^N(\lambda_1 \ge M) \le e^{7N/8} \int_M^\infty x^{N-1} e^{-\frac{N}{4}x^2}\,\mathrm{d}x\,. \tag{D.6e}$$

For $x \ge 8$, the logarithmic bound $\ln x \le x^2/30$ holds uniformly: the function $g(x) := x^2/30 - \ln x$ has $g'(x) = x/15 - 1/x > 0$ on $x \ge \sqrt{15}$, hence on $[8,\infty)$, and $g(8) = 64/30 - \ln 8 \approx 0.054 > 0$, so $g(x) \ge g(8) > 0$ throughout. This yields $x^{N-1} \le e^{\frac{N-1}{30}x^2} \le e^{\frac{N}{30}x^2}$, and bounds the integral by $\int_M^\infty e^{-Nx^2(1/4-1/30)}\mathrm{d}x = \int_M^\infty e^{-13Nx^2/60}\mathrm{d}x \le e^{-13NM^2/60}$ (valid for $NM \ge 60/26 \approx 2.31$, automatic under $N \ge 1$, $M \ge 8$). By a union bound across the $N$ eigenvalues and both signs (using $\sigma^N(\lambda_i \le -M) = \sigma^N(\lambda_1 \ge M)$ by the symmetry $\lambda \mapsto -\lambda$ of $\sigma^N$),

$$\sigma^N\Big(\max_{1\le i\le N} |\lambda_i| \ge M\Big) \le 2N\,\sigma^N(\lambda_1 \ge M) \le \exp\Big(-\frac{13NM^2}{60} + \frac{7N}{8} + \ln(2N)\Big)\,. \tag{D.6f}$$

To achieve the target bound of $e^{-NM^2/9}$, we require $M^2(\frac{13}{60}-\frac{1}{9}) \ge \frac{7}{8} + \frac{\ln(2N)}{N}$. Since $\frac{13}{60}-\frac{1}{9} = \frac{19}{180}$, evaluated at the constraint $M \ge 8$ (so $M^2 \ge 64$), the left-hand side is at least $64 \times \frac{19}{180} \approx 6.76$. This is strictly greater than $\frac{7}{8} + \frac{\ln(2N)}{N} \le \frac{7}{8} + \log 2 + \frac{1}{e} \approx 1.94$ for all $N \ge 1$. Hence the bound holds for all $M \ge 8$.

- The matrix $G_{d-1}$ drawn from the GOE$(d-1)$ in the Mehta normalization satisfies the distributional identity $J_{d-1} \overset{d}{=} \sqrt{2/(d-1)}\,G_{d-1}$. This establishes the exact spectral relation $\lambda_i(J_{d-1}) = \sqrt{2/(d-1)}\,\mu_i(G_{d-1})$. Applying the established bound with $N = d-1$ and substituting $M = t\sqrt{2/(d-1)}$, the threshold condition $M \ge 8$ translates identically to:

$$t\sqrt{\frac{2}{d-1}} \ge 8 \implies t \ge 4\sqrt{2(d-1)}\,. \tag{D.6g}$$

The exponential decay rate maps smoothly via:

$$-\frac{NM^2}{9} = -\frac{d-1}{9}\left(t\sqrt{\frac{2}{d-1}}\right)^2 = -\frac{2t^2}{9}\,. \tag{D.6h}$$

Mapping the probability events gives the required inequality:

$$\mathbb{P}\Big(\max_{1\le i\le d-1} |\mu_i(G_{d-1})| \ge t\Big) \le e^{-2t^2/9}\,. \tag{D.6i}$$

□

## D.4 Explicit Hermite tail integrals

**Lemma 14** (Explicit Hermite tail integrals)**.** *For every $u \in \mathbb{R}$ and integer $m \ge 1$ (the integration-by-parts identities below use only the finiteness of the boundary term $H_{m-1}(u)\,e^{-u^2/2}$ at the endpoint $u$ and the* *vanishing of the integrand at $+\infty$, so no positivity of $u$ is required),*

$$\int_u^\infty H_m(x)\, e^{-x^2/2}\,\mathrm{d}x \;=\; e^{-u^2/2} \sum_{k=0}^{\lfloor (m-1)/2\rfloor} 2^{k+1}\frac{(m-1)!!}{(m-2k-1)!!} H_{m-2k-1}(u) \;+\; R_m(u)\,, \tag{D.7a}$$

*where $R_m(u) = 0$ if $m$ is odd and $R_m(u) = 2^{m/2}(m-1)!!\int_u^\infty e^{-x^2/2}\,\mathrm{d}x$ if $m$ is even. The following companion identity (the special weight $e^{-x^2}$) is recorded for completeness and is* not used in the sequel*; the squared-Hermite tails entering $\delta_{\text{exact}}$ are the general-weight form $K_j^\beta$ of Corollary 11:*

$$\int_u^\infty H_j(x)^2\, e^{-x^2}\,\mathrm{d}x \;=\; e^{-u^2}\sum_{k=0}^{j-1} 2^k \frac{j!}{(j-k)!} H_{j-k-1}(u) H_{j-k}(u) + 2^j j! \int_u^\infty e^{-x^2}\,\mathrm{d}x\,.$$

*Proof.* • We evaluate $J_m = \int_u^\infty H_m(x) e^{-x^2/2}\,\mathrm{d}x$. We use two standard identities for the physicist Hermite polynomials:

$$H'_m(x) = 2m H_{m-1}(x) \tag{D.8a}$$

$$H_m(x) = 2xH_{m-1}(x) - 2(m-1)H_{m-2}(x) \tag{D.8b}$$

We begin by evaluating the integral of $2xH_{m-1}(x)e^{-x^2/2}$ using integration by parts. Let $f(x) = H_{m-1}(x)$ and $g'(x) = xe^{-x^2/2}$, which implies $g(x) = -e^{-x^2/2}$.

$$\begin{aligned}\int_u^\infty xH_{m-1}(x)e^{-x^2/2}\,\mathrm{d}x &= \Big[-H_{m-1}(x)e^{-x^2/2}\Big]_u^\infty + \int_u^\infty H'_{m-1}(x)e^{-x^2/2}\,\mathrm{d}x \\ &= H_{m-1}(u)e^{-u^2/2} + 2(m-1)\int_u^\infty H_{m-2}(x)e^{-x^2/2}\,\mathrm{d}x\end{aligned}$$

where we substituted (D.8a) in the second term. We then substitute $2xH_{m-1}(x)$ using a rearrangement of (D.8b), $2xH_{m-1}(x) = H_m(x) + 2(m-1)H_{m-2}(x)$, into the left-hand side:

$$\frac{1}{2}\int_u^\infty \Big(H_m(x) + 2(m-1)H_{m-2}(x)\Big)e^{-x^2/2}\,\mathrm{d}x = H_{m-1}(u)e^{-u^2/2} + 2(m-1)\int_u^\infty H_{m-2}(x)e^{-x^2/2}\,\mathrm{d}x$$

Distributing the integral and isolating $J_m = \int_u^\infty H_m(x)e^{-x^2/2}\,\mathrm{d}x$ yields the exact recurrence relation:

$$J_m = 2H_{m-1}(u)e^{-u^2/2} + 2(m-1)J_{m-2} \tag{D.8c}$$

By iteratively applying (D.8c), we can expand $J_m$:

$$\begin{aligned}J_m &= 2H_{m-1}(u)e^{-u^2/2} + 2(m-1)\Big[2H_{m-3}(u)e^{-u^2/2} + 2(m-3)J_{m-4}\Big] \\ &= e^{-u^2/2}\Big[2H_{m-1}(u) + 2^2(m-1)H_{m-3}(u) + 2^3(m-1)(m-3)H_{m-5}(u) + \dots\Big] + \text{Remainder}\end{aligned}$$

The $k$-th term of this expansion involves the cascading product $(m-1)(m-3)\cdots(m-2k+1)$, which evaluates precisely to the double factorial ratio $\frac{(m-1)!!}{(m-2k-1)!!}$. Incorporating the accumulated factor of $2^{k+1}$, the summation becomes:

$$e^{-u^2/2}\sum_{k=0}^{\lfloor (m-1)/2\rfloor} 2^{k+1}\frac{(m-1)!!}{(m-2k-1)!!}H_{m-2k-1}(u)$$

The recursion stops differently depending on parity:

- If $m$ is odd, the final evaluation step corresponds to $k = (m-1)/2$, meaning $H_0(u)$ is evaluated and no subsequent integrals are required. Thus, $R_m(u) = 0$.
- If $m$ is even, the final evaluated boundary term corresponds to $k = (m-2)/2$. The remaining integral falls back to $J_0 = \int_u^\infty H_0(x)e^{-x^2/2}\,\mathrm{d}x = \int_u^\infty e^{-x^2/2}\,\mathrm{d}x$. The accumulated multiplier in front of $J_0$ is $2^{m/2}(m-1)!!$, yielding exactly the prescribed $R_m(u)$.

- We evaluate $K_j = \int_u^\infty H_j(x)^2 e^{-x^2}\,\mathrm{d}x$. Consider the derivative of the product $H_{j-1}(x)H_j(x)e^{-x^2}$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[H_{j-1}(x)H_j(x)e^{-x^2}\right] = \left[H'_{j-1}(x)H_j(x) + H_{j-1}(x)H'_j(x) - 2xH_{j-1}(x)H_j(x)\right]e^{-x^2}$$

Substitute the derivative identity (D.8a) for the first two terms:

$$\left[2(j-1)H_{j-2}(x)H_j(x) + 2jH_{j-1}(x)^2 - 2xH_{j-1}(x)H_j(x)\right]e^{-x^2}$$

Using (D.8b), we can replace $2xH_{j-1}(x)$ with $H_j(x) + 2(j-1)H_{j-2}(x)$ in the last term. This gives:

$$2(j-1)H_{j-2}(x)H_j(x) + 2jH_{j-1}(x)^2 - \Big(H_j(x) + 2(j-1)H_{j-2}(x)\Big)H_j(x)$$

The cross-terms $2(j-1)H_{j-2}(x)H_j(x)$ cancel, leaving:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[H_{j-1}(x)H_j(x)e^{-x^2}\right] = \Big(2jH_{j-1}(x)^2 - H_j(x)^2\Big)e^{-x^2}$$

Integrating this equation from $u$ to $\infty$ yields:

$$\left[H_{j-1}(x)H_j(x)e^{-x^2}\right]_u^\infty = 2j\int_u^\infty H_{j-1}(x)^2 e^{-x^2}\,\mathrm{d}x - \int_u^\infty H_j(x)^2 e^{-x^2}\,\mathrm{d}x$$

Because the exponential $e^{-x^2}$ vanishes at $\infty$, evaluating the bounds leaves:

$$-H_{j-1}(u)H_j(u)e^{-u^2} = 2jK_{j-1} - K_j$$

Rearranging this isolates our desired recurrence for $K_j$:

$$K_j = H_{j-1}(u)H_j(u)e^{-u^2} + 2jK_{j-1} \tag{D.8d}$$

By unrolling (D.8d) successively $j$ times, we extract boundary terms $H_{j-k-1}(u)H_{j-k}(u)e^{-u^2}$ at each step $k$ (where $k$ ranges from 0 to $j-1$). At step $k$, the initial multiplier accumulates as $2j\cdot 2(j-1)\cdots 2(j-k+1)$, which simplifies directly to $2^k\frac{j!}{(j-k)!}$. This gives the exact finite sum provided in the lemma statement. After $j$ steps, the recursion completely bottoms out at $K_0 = \int_u^\infty H_0(x)^2 e^{-x^2}\,\mathrm{d}x = \int_u^\infty e^{-x^2}\,\mathrm{d}x$. The accumulated multiplier in front of $K_0$ is $2^j j!$, yielding the exact stated remainder. □

The next lemma extends Lemma 14 from the canonical Gaussian weight $e^{-y^2/2}$ to an arbitrary weight $e^{-\beta y^2/2}$ with $\beta > 0$. The general $\beta$-version gives the exact squared-Hermite tail $T_d^{\mathrm{exact}}(u)$ used in Theorem 4: when integrating $H_j(\rho x)^2$ against $e^{-(1+\rho^2)x^2/2}$ on $[u,\infty)$ and substituting $y = \rho x$, the relevant exponent is $\beta = (1+\rho^2)/\rho^2 = (3k-2)/k > 2$ for $k \geq 3$, a regime to which the existing Lemma 14 ($\beta = 1$) does not apply.

**Lemma 15** (Generalized Hermite–Gaussian tail recurrence)**.** *For every $\beta > 0$, every integer $m \geq 0$, and every $u \in \mathbb{R}$, set*

$$J_m^\beta(u) \;:=\; \int_u^\infty H_m(y)\,e^{-\beta y^2/2}\,\mathrm{d}y\,. \tag{D.9a}$$

*Write $\gamma := 2/\beta$ and $\theta := (2-\beta)/\beta$. The base cases are*

$$J_0^\beta(u) \;=\; \sqrt{\tfrac{2\pi}{\beta}}\,\bar{\Phi}(u\sqrt{\beta})\,, \qquad J_1^\beta(u) \;=\; \gamma\,e^{-\beta u^2/2}\,, \tag{D.9b}$$

*and for every $m \geq 2$, the family $\{J_m^\beta\}$ satisfies the three-term recurrence*

$$J_m^\beta(u) \;=\; \gamma\,H_{m-1}(u)\,e^{-\beta u^2/2} \;+\; 2(m-1)\,\theta\,J_{m-2}^\beta(u)\,. \tag{D.9c}$$

*Iterating (D.9c) yields the explicit closed form*

$$J_m^\beta(u) \;=\; \gamma\,e^{-\beta u^2/2}\sum_{k=0}^{\lfloor (m-1)/2\rfloor}(2\theta)^k\,\frac{(m-1)!!}{(m-2k-1)!!}\,H_{m-2k-1}(u) \;+\; R_m^\beta(u)\,, \tag{D.9d}$$

*where $R_m^\beta(u) = 0$ if $m$ is odd, and $R_m^\beta(u) = (2\theta)^{m/2}(m-1)!!\,J_0^\beta(u)$ if $m$ is even.*

For $\beta = 1$, $\gamma = 2$ and $\theta = 1$, so $\gamma(2\theta)^k = 2^{k+1}$, and Lemma 15 reduces exactly to (D.7a); the remainder $R^\beta_m$ specialises to $2^{m/2}(m-1)!!\int_u^\infty e^{-y^2/2}\,dy$, matching the existing $R_m$. For $\beta > 2$, $\theta < 0$, and the iterated sum (D.9d) carries alternating signs in $k$ but remains exact pointwise; in subsequent estimates one must not pass to absolute values of individual summands.

*Proof of Lemma 15.* • Apply integration by parts with $f(y) = H_{m-1}(y)$ and $g'(y) = 2y\,e^{-\beta y^2/2}$, hence $g(y) = -(2/\beta)\,e^{-\beta y^2/2} = -\gamma\,e^{-\beta y^2/2}$. The boundary term at $+\infty$ vanishes by Gaussian decay, and using $H'_{m-1}(y) = 2(m-1)\,H_{m-2}(y)$,

$$\int_u^\infty 2y\,H_{m-1}(y)\,e^{-\beta y^2/2}\,dy \;=\; \gamma\,H_{m-1}(u)\,e^{-\beta u^2/2} \;+\; 2(m-1)\,\gamma\,J^\beta_{m-2}(u)\,.$$

Combining with the Hermite recurrence $H_m(y) = 2y\,H_{m-1}(y) - 2(m-1)\,H_{m-2}(y)$,

$$J^\beta_m(u) \;=\; \int_u^\infty 2y\,H_{m-1}(y)\,e^{-\beta y^2/2}\,dy \;-\; 2(m-1)\,J^\beta_{m-2}(u)\,,$$

and $2(m-1)\gamma - 2(m-1) = 2(m-1)(\gamma-1) = 2(m-1)\theta$, yielding (D.9c).

• For the boundary cases, $J^\beta_0(u) = \int_u^\infty e^{-\beta y^2/2}\,dy = \sqrt{2\pi/\beta}\,\bar\Phi(u\sqrt{\beta})$ by direct change of variable, and $J^\beta_1(u) = \int_u^\infty 2y\,e^{-\beta y^2/2}\,dy = (2/\beta)\,e^{-\beta u^2/2} = \gamma\,e^{-\beta u^2/2}$.

• Unrolling (D.9c) $k$ times and using the telescoping product $(m-1)(m-3)\cdots(m-2k+1) = (m-1)!!/(m-2k-1)!!$,

$$J^\beta_m(u) \;=\; \gamma\,e^{-\beta u^2/2}\sum_{j=0}^{k-1}(2\theta)^j\,\frac{(m-1)!!}{(m-2j-1)!!}\,H_{m-2j-1}(u) \;+\; (2\theta)^k\,\frac{(m-1)!!}{(m-2k-1)!!}\,J^\beta_{m-2k}(u)\,.$$

For $m$ odd, take $k = (m-1)/2$: $m-2k-1 = 0$, hence $(m-2k-1)!! = 0!! = 1$, the residual factor $J^\beta_{m-2k}(u) = J^\beta_1(u) = \gamma\,e^{-\beta u^2/2}$, and the residual term coincides with the $j = (m-1)/2$ summand of the main sum (since $H_{m-2k-1}(u) = H_0(u) = 1$ and $\gamma e^{-\beta u^2/2} = J^\beta_1(u)$); absorbing it into the main sum gives $R^\beta_m = 0$. For $m$ even, take $k = m/2$: $m-2k-1 = -1$, $(m-2k-1)!! = (-1)!! = 1$, the residual factor is $J^\beta_0(u)$, and $R^\beta_m(u) = (2\theta)^{m/2}\,(m-1)!!\,J^\beta_0(u)$.

• *Boundary verification ($m = 1, 2$).* For $m = 1$, the sum (D.9d) reduces to $\gamma\,e^{-\beta u^2/2}\,H_0(u) = \gamma\,e^{-\beta u^2/2}$, matching $J^\beta_1(u)$. For $m = 2$, the main sum is $\gamma\,e^{-\beta u^2/2}\,H_1(u) = 2\gamma u\,e^{-\beta u^2/2}$ and $R^\beta_2(u) = 2\theta\,J^\beta_0(u)$; this matches the direct integration $J^\beta_2(u) = \int_u^\infty(4y^2-2)e^{-\beta y^2/2}\,dy = 2\gamma u\,e^{-\beta u^2/2} + 2\theta\,J^\beta_0(u)$ obtained via integration by parts. ☐

**Corollary 11** (Squared-Hermite Gaussian tail)**.** *For every $\beta > 0$, every integer $j \geq 0$, and every $u \in \mathbb{R}$, the squared-Hermite tail*

$$K^\beta_j(u) \;:=\; \int_u^\infty H_j(y)^2\,e^{-\beta y^2/2}\,dy \tag{D.10a}$$

*admits the exact closed form*

$$K^\beta_j(u) \;=\; \sum_{p=0}^{j} 2^p\,p!\binom{j}{p}^2 J^\beta_{2j-2p}(u)\,, \tag{D.10b}$$

*where each $J^\beta_{2j-2p}$ is given in closed form by Lemma 15.*

*Proof of Corollary 11.* The classical linearization of the product of two physicist Hermite polynomials reads

$$H_m(y)\,H_n(y) \;=\; \sum_{p=0}^{\min(m,n)} 2^p\,p!\binom{m}{p}\binom{n}{p}\,H_{m+n-2p}(y)\,.$$

Specialising to $m = n = j$ yields $H_j(y)^2 = \sum_{p=0}^{j} 2^p\,p!\,\binom{j}{p}^2\,H_{2j-2p}(y)$. Multiplying by $e^{-\beta y^2/2}$, integrating from $u$ to $\infty$, and exchanging the finite sum with the integral (justified termwise since each $H_m\,e^{-\beta y^2/2}$ is absolutely integrable on $[u,\infty)$), one obtains (D.10b). ☐

# E List of notations

### Tensor regression model

| | |
|---|---|
| $k \geq 2$ | Order of the symmetric tensors (all theorems require $k, d \geq 3$). |
| $d \geq 2$ | Ambient dimension of the underlying vectors (tensors have $d^k$ entries). |
| $n = d - 1$ | Effective dimension (associated to the sphere and the GOE matrices). |
| $R \geq 1$ | Rank of the signal tensor $\boldsymbol{\sigma}^\star$ (and rank bound for the candidate tensors $\boldsymbol{\sigma}$). |
| $\kappa \in (0, 1]$ | Coherence parameter. |
| $\lambda > 0$ | Signal-to-noise ratio. |
| $\boldsymbol{Y}, \boldsymbol{W}, \boldsymbol{\sigma}^\star$ | Observed tensor, standard Gaussian noise tensor, and planted (true) normalized signal tensor $\sigma^\star$, respectively. |
| $\boldsymbol{\sigma}$, $\hat{\boldsymbol{\sigma}}$ | Generic candidate tensor and the profile MLE (1.3). |
| $\mathcal{C}_{R,\kappa}$, $\boldsymbol{G}$ | Feasible set $\{\boldsymbol{\sigma} \in \mathfrak{S}_R : \kappa(\boldsymbol{\sigma}) \geq \kappa\}$; coherence $\kappa(\cdot)$ from (1.2b) via the Gram matrix $\boldsymbol{G}$, $G_{ij} = \langle t_i, t_j \rangle^k$. |
| $\langle \cdot, \cdot \rangle_{\mathcal{T}}, \| \cdot \|_F$ | Tensor canonical inner product and Frobenius norm. |
| $\mathcal{T}(k, d)$ | Space of symmetric tensors of order $k$ and dimension $d$. |
| $\mathbb{S}(k, d)$ | Unit sphere in $\mathcal{T}(k, d)$. |
| $\mathfrak{S}_R$ | Set of symmetric normalized tensors of rank at most $R$. |

### Geometry and Random Fields

| | |
|---|---|
| $\mathbb{S}^{d-1}$ | Unit sphere in $\mathbb{R}^d$, which has dimension $n$. |
| $\lvert\mathbb{S}^{d-1}\rvert$ | Surface area of the unit sphere, $\lvert\mathbb{S}^{d-1}\rvert = 2\pi^{d/2}/\Gamma(d/2)$. |
| $T_{\boldsymbol{\theta}}\mathbb{S}^{d-1}$ | Tangent space to the sphere at point $\boldsymbol{\theta}$, also of dimension $n$. |
| $X(\boldsymbol{\theta})$ | Centered Gaussian random field defined on the sphere by $X(\boldsymbol{\theta}) = \langle \boldsymbol{W}, \boldsymbol{\theta}^{\otimes k} \rangle_{\mathcal{T}}$ (the Kostlan–Shub–Smale random field). |
| $\nabla X, \nabla^2 X$ | Riemannian gradient and Riemannian Hessian of the random field $X$. |
| $\Gamma_{R,\kappa}$ | Supremum of the noise over the manifold constraints defined by $\mathfrak{S}_R$ and $\kappa$. |
| $\Gamma_{1,1}$ | Two-sided absolute supremum of the field, $\Gamma_{1,1} = \sup_{\boldsymbol{\theta}} \lvert X(\boldsymbol{\theta})\rvert$. |
| $\widetilde{\Gamma}_{1,1}(u)$ | One-sided excursion probability $\widetilde{\Gamma}_{1,1}(u) = \mathbb{P}\{\sup_{\boldsymbol{\theta}} X(\boldsymbol{\theta}) > u\}$, related to $\Gamma_{1,1}$ by $\mathbb{P}\{\Gamma_{1,1} > u\} \leq 2\,\widetilde{\Gamma}_{1,1}(u)$. |

### Random Matrices and Spectra

| | |
|---|---|
| $G_{d-1}$ | Random $n \times n$ matrix drawn from the Gaussian Orthogonal Ensemble (GOE). |
| $\mu_1, \ldots, \mu_{d-1}$ | Real eigenvalues of the matrix $G_{d-1}$. |
| $M_{d-1}$ | Spectral radius of $G_{d-1}$, $M_{d-1} = \max_i \lvert\mu_i\rvert$. |
| $\rho$ | Scaling constant appearing for the GOE shift, defined as $\rho = \sqrt{k/(2(k-1))}$. |
| $I_{d-1}$ | Identity matrix of dimension $n$. |

### Special Functions and Analysis

| | |
|---|---|
| $\bar{\Phi}(u)$ | Standard Gaussian tail, $\bar{\Phi}(u) = \mathbb{P}(Z > u)$. |
| $\varphi(x)$ | Density function of the standard normal distribution, $\varphi(x) = (2\pi)^{-1/2}e^{-x^2/2}$. |
| $\Gamma(\cdot)$ | Gamma function. |
| $H_j(x)$ | Hermite polynomial of degree $j$ (physicists' convention, $H_n(x) \sim (2x)^n$). |
| $q_d, Q_d$ | Weighted and unweighted characteristic polynomials appearing in the Fyodorov–Mehta orthogonal expansion. |
| $u$ | Threshold level for bounding the exceedance probability. |
| $I_m(u; a)$ | Incomplete variance integral, $I_m(u; a) = \int_u^\infty x^m e^{-ax^2/2}\,dx$. |

**Kac–Rice integral, tail bounds, and thresholds**

| | |
|---|---|
| $C_{k,d}$ | Kac–Rice prefactor, $C_{k,d} = 2\sqrt{\pi}\,(k-1)^{(d-1)/2}/\Gamma(d/2)$. |
| $\delta_0(u)$ | Kac–Rice integral (1.4d) bounding $\widetilde{\Gamma}_{1,1}(u)$. |
| $\delta_{\mathrm{bl}}(u)$ | Asymptotic baseline (1.5); $\delta_0 \sim \delta_{\mathrm{bl}}$ as $u \to \infty$ (not a bound). |
| $\delta_{\mathrm{exact}}(u)$ | Exact closed-form evaluation of $\delta_0$ (Theorem 2). |
| $\delta_{\mathrm{IMF}},\ \delta^\star_{\mathrm{IMF}}$ | Improved Mehta–Fyodorov bound and its sharpened variant (Theorems 3, 4). |
| $\delta_{\mathrm{SMF}},\ \delta_{\mathrm{SM}}$ | Simplified Mehta–Fyodorov and spectral-method bounds (Theorems 5, 6). |
| $\delta_{\min}(u)$ | Master failure probability $\delta_{\min} = 2\,\delta_{\mathrm{IMF}}$ (Theorem 1). |
| $u_{\mathrm{IMF}}, u_{\mathrm{SMF}}, u_{\mathrm{SM}}$ | Validity thresholds $\sqrt{2d-1}/\rho$, $2\sqrt{d}$, $32\sqrt{d-1}/\rho$. |
| $u_\alpha,\ \alpha_0(k,d)$ | Confidence-level inversion (1.8) and its validity threshold (Remark 1). |

**Mehta–Fyodorov constants and Hermite-tail integrals**

| | |
|---|---|
| $\Lambda,\ \beta$ | $\Lambda = 2\rho^2 - 1 = 1/(k-1)$; $\beta = (1+\rho^2)/\rho^2 = (3k-2)/k$. |
| $c_j,\ \mu_m$ | $c_j = (2^j j!\sqrt{\pi})^{-1/2}$; $\mu_m = \int_{\mathbb{R}} H_m(y)e^{-y^2/2}\,dy$. |
| $\alpha_d,\ \beta_d$ | Dominant Hermite coefficient and remainder envelope in $Q_d = \alpha_d H_{d-1} + \mathcal{R}_d$ (Lemma 5). |
| $\Phi_d,\ \Psi_d$ | Polynomial–rational functions of the IMF decomposition (Proposition 2). |
| $\eta_d(\rho, u)$ | Layer-cake correction of the SM bound (Proposition 4). |
| $\mathcal{I}^c_d(\nu)$ | Hermite tail integral $\int_\nu^\infty H_d(y)e^{-y^2/2}\,dy$. |
| $\mathcal{L}_d,\ \mathcal{C}_d$ | Linear Hermite tail and cross integral in the $\delta_{\mathrm{exact}}$ decomposition. |
| $D_1, \ldots, D_4$ | The four pieces of $\delta_{\mathrm{exact}}$ (Theorem 2). |
| $J^\beta_m,\ K^\beta_j,\ T^{\mathrm{exact}}_d$ | Generalised Hermite and squared-Hermite tail integrals (Lemma 15, Corollary 11). |

**Spherical $k$-spin complexity (Section 4)**

| | |
|---|---|
| $N^{\mathrm{lm}}_{[E,\infty)},\ N^{\mathrm{cp}}_{[E,\infty)}$ | Number of local maxima, resp. critical points, of $X$ with value $\geq E$. |
| $E,\ e$ | Energy level and reduced energy $e = \rho E/\sqrt{2(d-1)}$. |
| $E_\infty,\ E_0$ | ABC spectral-edge energy $2\sqrt{(k-1)/k}$ and ground-state energy $E_0 > E_\infty$. |
| $E_{\mathrm{BDG}},\ u_{\mathrm{ABC}}$ | Two-sided-bracket threshold $8\sqrt{2(d-1)}/\rho$; ABC energy scale $u_{\mathrm{ABC}} = E/\sqrt{d}$. |
| $\Theta_k,\ \Theta_p,\ \Theta_{0,p}$ | ABC complexity functions: local-maximum rate of the $k$-spin field, and total resp. index-0 rates of the $p$-spin model (Corollary 9). |
| $C_{\mathrm{amp}},\ \delta_{\mathrm{BDG}}$ | Amplifier constant $8\sqrt{2}$ and GOE spectral-radius bound $\delta_{\mathrm{BDG}}(\rho E) = e^{-2(\rho E)^2/9}$ (Lemma 13). |