Non-asymptotic First- and Second-Order Tail Brackets for the Kostlan–Shub–Smale Field and Spherical k-Spin Complexity

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Abstract

For the Kostlan–Shub–Smale field X of degree $k \geq 3$ on the sphere $\mathbb{S}^{d-1}$, we prove the two-sided bound $(1-\varepsilon(u))\,\delta_*(u) \leq \mathbb{P}\{\sup X > u\} \leq \delta_*(u)$ at every finite (k,d), where $\delta_*(u)$ is the expected number of critical points above u, halved when k is even, and $\varepsilon(u)$ is explicit for $\rho u \geq \min\{d-1, \frac{3}{2}\sqrt{2(d-1)}\}$. The gap is therefore at most of order $e^{-(1+\rho^2)u^2/2}$ with $\rho^2 = k/(2(k-1))$. We also give δ_* itself in closed form, as a finite sum of Hermite polynomials, together with a two-term envelope that is asymptotically sharp and a single-monomial relaxation of the form $Cu^{d-2}e^{-u^2/2}$. As an application we bracket the annealed complexity of the spherical k-spin Hamiltonian at every finite (k,d): the upper bound holds at every energy, a lower bound of the same form at every energy above $9/4$ times the spectral edge $\sqrt{2(d-1)}/\rho$, above twice the edge when $d \geq 10$, and is consistent with the Auffinger–Ben Arous–Černý complexity function as $d \to \infty$.

Keywords. Kostlan–Shub–Smale field; Kac–Rice formula; tail of the supremum of a Gaussian field; Gaussian Orthogonal Ensemble; spherical k-spin model; annealed complexity.
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1 Introduction

Let X be the Kostlan–Shub–Smale (KSS) field of degree $k \geq 3$ on the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$, $d \geq 3$: the centered Gaussian field with covariance $\mathbb{E}[X(s)X(t)] = \langle s,t\rangle^k$. It is the canonical isotropic Gaussian homogeneous polynomial of degree k, the Hamiltonian of the spherical k-spin model up to normalization, and the noise part of the spiked tensor model; its paths are polynomials. This paper is about one quantity, $\mathbb{P}\{\sup_{\mathbb{S}^{d-1}} X > u\}$, and about how well the Kac–Rice first moment describes it at finite (k,d).

Two things are proved. The first is that the Kac–Rice bound is not only an upper bound but a two-sided one, with an explicit error: Theorem 1.1 brackets the supremum probability between $(1-\varepsilon(u))\,\delta_*(u)$ and $\delta_*(u)$, where $\delta_*(u)$ is the expected number of critical points above u (halved for k even) and $\varepsilon(u)$ is explicit at every finite (k,d). The second is that δ_* itself is computable: it has a closed form as a finite sum of Hermite polynomials (Theorem 2.4), together with two explicit relaxations (Theorems 2.7 and 2.10)

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that trade tightness for a form one can invert. Section 5 applies both to the annealed complexity of the spherical k-spin Hamiltonian.

The point of the bracket is that it turns a bound into an approximation. Without it, an explicit upper bound on a tail can always be suspected of being loose by an unknown factor, and there is no baseline against which to judge it. Here the baseline is δ_* itself, and the bracket says the first moment is right to within $\varepsilon(u)$, which decays like $e^{-\rho^2 u^2/2}$ with $\rho^2 = k/(2(k-1))$.

1.1 The Kac–Rice integral and the conditional Hessian

Throughout, $\rho := \sqrt{k/(2(k-1))}$ and $\varsigma^2 := 1/\rho^2 = 2(k-1)/k$; Φ and φ are the standard Gaussian distribution function and density, $\bar\Phi := 1-\Phi$; $\lambda_{\max}$ is the largest eigenvalue of a symmetric matrix, $\|G\|$ the operator norm of a matrix G, and $\|Y\|_p := (\mathbb{E}|Y|^p)^{1/p}$ for a random variable Y; $N^{\mathrm{cp}}_{[u,\infty)}(F)$ and $N^{\mathrm{lm}}_{[u,\infty)}(F)$ count the critical points and the local maxima of a smooth function F with value $\ge u$, the argument being omitted when $F = X$. The Gaussian Orthogonal Ensemble of size n, GOE(n), is the law of a real symmetric $n\times n$ random matrix $G = (G_{ij})$ whose entries are jointly Gaussian, centered, independent up to symmetry, with the normalization $\mathrm{Var}(G_{ii}) = 1$ and $\mathrm{Var}(G_{ij}) = 1/2$ for $i \ne j$; equivalently, the density on the space of real symmetric matrices is proportional to $\exp(-\mathrm{tr}(G^2)/2)$ and the law is invariant under conjugation by any orthogonal matrix [14].

Fix $t \in \mathbb{S}^{d-1}$ and an orthonormal basis $(e_a)_{a\le d-1}$ of $t^\perp = T_t\mathbb{S}^{d-1}$. Differentiating $\langle s,t\rangle^k$, the covariant derivatives of X at t satisfy

$$\begin{aligned}
&\mathrm{Cov}(\partial_a X(t), \partial_b X(t)) = k\delta_{ab}, \qquad \mathrm{Cov}(X(t), \nabla X(t)) = 0, \qquad \mathrm{Cov}(\nabla X(t), \nabla^2 X(t)) = 0,\\
&\mathrm{Cov}(X(t), \nabla^2_{ab} X(t)) = -k\delta_{ab}, \qquad \mathrm{Cov}(\nabla^2_{ab} X(t), \nabla^2_{cd} X(t)) = k(k-1)(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) + k^2\delta_{ab}\delta_{cd}.
\end{aligned}$$

Hence the matrix

$$G := \frac{\nabla^2 X(t) + k\, X(t)\, I_{d-1}}{\sqrt{2k(k-1)}} \;\sim\; \mathrm{GOE}(d-1) \tag{1.1a}$$

is independent of $(X(t), \nabla X(t))$, the gradient $\nabla X(t) \sim N(0, kI_{d-1})$ is independent of $(X(t), \nabla^2 X(t))$, and conditionally on $\{X(t) = x,\ \nabla X(t) = 0\}$,

$$\nabla^2 X(t) = \sqrt{2k(k-1)}\,\big(G - \rho x\, I_{d-1}\big), \tag{1.1b}$$

since $\sqrt{2k(k-1)}\,\rho = k$; this is the conditional law of Auffinger et al. [2, Lemma 3.2(b)]. The critical point t is a nondegenerate local maximum if and only if $\lambda_{\max}(G) < \rho x$. The Kac–Rice formula for the critical points of X [1, Theorem 6.1], with $p_{\nabla X(t)}(0) = (2\pi k)^{-(d-1)/2}$ and (1.1b), gives

$$\begin{aligned}
\delta_{\mathrm{exact}}(u) &:= \mathbb{E}\, N^{\mathrm{cp}}_{[u,\infty)} = C_{k,d}\int_u^\infty A(\rho x)\,\varphi(x)\,\mathrm{d}x,\\
A(\nu) &:= \mathbb{E}\big|\det(G - \nu I_{d-1})\big|, \qquad C_{k,d} := \mathrm{Vol}(\mathbb{S}^{d-1})\Big(\frac{k-1}{\pi}\Big)^{\frac{d-1}{2}} = \frac{2\sqrt{\pi}}{\Gamma(d/2)}\,(k-1)^{\frac{d-1}{2}},
\end{aligned} \tag{1.1c}$$

an equality for the expected number of critical points above u, which Theorem 2.4 computes in closed form. For k even, $X(-s) = X(s)$ and the critical points come in antipodal pairs. Since the maximum of X over the compact $\mathbb{S}^{d-1}$ is attained at a critical point, $\mathbf{1}\{\sup X > u\} \le N^{\mathrm{cp}}_{[u,\infty)}$, and taking expectations

$$\mathbb{P}\Big\{\sup_{\mathbb{S}^{d-1}} X > u\Big\} \;\le\; \delta_*(u) \;:=\; \frac{2}{3+(-1)^k}\,\delta_{\mathrm{exact}}(u)\,. \tag{1.1d}$$

The bound is loose by the critical points above u that are not the global maximum, saddle points and beaten local maxima alike; Theorem 1.1 shows that this slack is a factor $\varepsilon(u) = O(u^{d-2}e^{-\rho^2u^2/2})$. The bound (1.1d) is sharp at second order. For a Gaussian field of constant variance on a parameter set with

bounded curvature, Azaïs and Wschebor [5, Theorem 8.12] prove that the Rice bound on the density of the maximum exceeds that density by at most $\exp(-(1+\gamma)u^2/2+o(u^2))$, with $\gamma>0$ given by the conditional variance of the field at two points (and a boundary term, absent here). The scale δ_{exact} is therefore the right one for the supremum probability as $u\to\infty$; the constants of that theorem are asymptotic in the level, and Theorem 1.1 gives the matching lower bound at finite (k,d) by another route.

Section 2 evaluates δ_{exact} in closed form for every u (Theorem 2.4) and gives $\delta_{\mathrm{IMF}}\geq\delta_{\mathrm{exact}}$ on $[u_{\mathrm{IMF}},\infty)$, with $u_{\mathrm{IMF}}:=\sqrt{2d-1}/\rho$, and $\delta_{\mathrm{SMF}}\geq\delta_{\mathrm{IMF}}$ on $[u_{\mathrm{SMF}},\infty)$ (Proposition 2.12).

Asymptotic baseline as $u\to\infty$, with d fixed. Replacing $A(\rho x)$ in (1.1c) by its leading monomial $(\rho x)^{d-1}$ gives, by Lemma B.2, the asymptotic equivalent

$$\delta_{\mathrm{bl}}(u)\;:=\;\frac{\sqrt{2}}{\Gamma(d/2)}\Big(\frac{k}{2}\Big)^{\frac{d-1}{2}}u^{d-2}\,e^{-u^2/2}\,,\qquad \delta_{\mathrm{exact}}(u)\;\sim\;\delta_{\mathrm{bl}}(u)\quad\text{as } u\to\infty \text{ with } d \text{ fixed}\,. \tag{1.2}$$

This is the optimal tail rate against which the non-asymptotic bounds are measured; δ_{bl} is not itself an upper bound on δ_{exact}, only the asymptotic equivalent.

Constants. The constants α_d,β_d and the functions Φ_d,Ψ_d of Table 1 are defined in (2.10c), (2.10a) and (2.7b)–(2.7c); all are explicit at finite (k,d).

Method	Validity range	Closed-form expression
Baseline (not a bound)	$u\to\infty$, d fixed	$\delta_{\mathrm{bl}}(u)\;=\;\frac{\sqrt{2}}{\Gamma(d/2)}\Big(\frac{k}{2}\Big)^{\frac{d-1}{2}}u^{d-2}e^{-u^2/2}$
δ_{exact} (canonical)	$u\in\mathbb{R}$ (exact)	$\delta_{\mathrm{exact}}(u)\;=\;2(k-1)^{\frac{d-1}{2}}\big[D_1(u)+D_2(u)+D_3(u)+D_4(u)\big]$, finite Hermite sum (Thm. 2.4)
δ_{IMF}	$u\geq u_{\mathrm{IMF}}$	$\delta_{\mathrm{IMF}}(u)\;=\;2(k-1)^{\frac{d-1}{2}}\Big(\alpha_d\,\Phi_d(\rho,u)\,e^{-u^2/2}+\Psi_d(\rho,u)\,e^{-(1+\rho^2)u^2/2}\Big)$
δ_{SMF}	$u\geq u_{\mathrm{SMF}}$	$\delta_{\mathrm{SMF}}(u)\;=\;4\,\alpha_d(2k)^{\frac{d-1}{2}}u^{d-2}e^{-u^2/2}+2^d\beta_d(k-1)^{\frac{d-1}{2}}u^{2d-3}e^{-3u^2/4}$

Table 1: The Kac–Rice integral $\delta_{\mathrm{exact}}(u)$ in closed form and its two explicit non-asymptotic relaxations, with the asymptotic baseline δ_{bl} in the first row, which is the equivalent $\delta_{\mathrm{exact}}\sim\delta_{\mathrm{bl}}$ as $u\to\infty$ at fixed d and not an upper bound. The exact form δ_{exact} is the Kac–Rice integral itself, evaluated in closed form for every u; the threshold u_{IMF} marks the range on which $\delta_{\mathrm{exact}}\leq\delta_{\mathrm{IMF}}$. The constants α_d,β_d and the functions Φ_d,Ψ_d are defined in (2.10c), (2.10a) and (2.7b)–(2.7c).

1.2 The main result

For $t\in\mathbb{S}^{d-1}$, let G be the matrix (1.1a) and ζ^t the helix process of Section 3, the normalized residual of the regression of X on $(X(t),\nabla X(t))$, defined on $M=\mathbb{S}^{d-1}\setminus\{\pm t\}$; put

$$\Xi(x):=\mathbb{E}\Big[\big|\det(\rho xI-G)\big|\;\mathbb{P}\Big(\sup_M\zeta^t>x\;\Big|\;G\Big)\Big], \tag{1.3}$$

which does not depend on t by rotational invariance. Lemma 3.1 and Proposition 3.2 are stated for k odd; Sections 3.1 to 4.4 hold for both parities, and Section 4.5 treats k even.

Theorem 1.1 (Non-asymptotic second-order bracket)**.** *Let $k \geq 3$ and $d \geq 3$. For every $u > 0$,*

$$\big(1-\varepsilon(u)\big)\,\delta_*(u) \;\leq\; \mathbb{P}\Big\{\sup_{\mathbb{S}^{d-1}} X > u\Big\} \;\leq\; \delta_*(u), \qquad \varepsilon(u) := \sup_{x\geq u} \frac{\Xi(x)}{A(\rho x)} \in [0,1],$$

with Ξ as in (1.3), and for $\rho u \geq d-1$

$$\begin{gathered} \varepsilon(u) \;\leq\; C^{\mathrm{gap}}_{k,d}\,\frac{(1+\rho u)^{d-1}}{\rho u}\,e^{-\rho^2u^2/2}, \\ C^{\mathrm{gap}}_{k,d} = 4\Big(2+\tfrac{5}{6}\sqrt{d-1}\Big)^{d-1} C^{\mathrm{Rice}}_{k,d}, \qquad C^{\mathrm{Rice}}_{k,d} = c_k\,\frac{\sqrt{\pi}\;k^3\;2^{2k}\;(k-1)^{(d-2)/2}}{\Gamma\big(\frac{d-1}{2}\big)}, \\ c_k = 43 \text{ for } k \text{ odd}, \qquad c_k = \tfrac{9}{2} \text{ for } k \text{ even}. \end{gathered} \tag{1.4}$$

Remark 1.2. For $\frac{3}{2}\sqrt{2(d-1)} \leq \rho u < d-1$, a non-empty range when $d \geq 6$, the bound (1.4) holds with the factor 4 of $C^{\mathrm{gap}}_{k,d}$ replaced by $\frac{15}{4}\big(\frac{9}{5}\big)^{(d-1)/2}$.

Hence $\delta_*(u) - \mathbb{P}\{\sup X > u\}$ decays at least like $e^{-(1+\rho^2)u^2/2}$, with $1+\rho^2 = 1+k/(2(k-1))$ the second-order exponent given by Azaïs and Wschebor [5, Theorem 8.12], here with explicit finite-u constants.

Strategy. The first half of the proof (Lemma 3.1 to Corollary 3.3) writes the gap as the expected number of critical points above u that are beaten by a larger value of X. Conditionally on $X(t) = x$ and $\nabla X(t) = 0$, the field at polar angle θ from t is $X(s) = \cos^k\theta\, x + (1-\cos^k\theta)\,\zeta^t(s)$, where ζ^t is the *helix process*, the normalized residual of the regression of X on $(X(t), \nabla X(t))$; so t is beaten if and only if $\sup\zeta^t > x$, the gap equals $C_{k,d}\int_u^\infty \Xi(x)\varphi(x)\,\mathrm{d}x$ with $\Xi(x) = \mathbb{E}[|\det(\rho x I - G)|\,\mathbb{P}(\sup\zeta^t > x \mid G)]$, and Hölder's inequality with the exponent $p = \rho^2x^2$ separates the weight, an L^p norm of the GOE controlled by the Gaussian tail of $\|G\|$, from the tail of $\sup\zeta^t$, whose power $1-1/p$ costs only a constant because the fiber value $\varsigma\lambda_{\max}(G)$ alone gives the lower bound $\mathbb{P}\{\sup\zeta^t > x\} \geq \bar\Phi(\rho x)$. The second half (Sections 3.1 to 4.4) bounds that tail by the Rice formula. The helix process is a finite sum of independent KSS fields of lower degree on the equatorial sphere, the layers, with weights depending only on θ; its variance is at most ς^2, with equality only at the pole, where it reduces to $\varsigma v^\top G v$. Its supremum is attained at an interior local maximum, never on the pole fibers, so $\mathbb{P}\{\sup\zeta^t > x\}$ is at most the expected number of local maxima above x. In the Kac–Rice integral for that number, the Gaussian density at polar angle θ has exponent at least $y^2/(2\varsigma^2)$, and the diagonal of the conditional Hessian consists of independent Gaussian variables, so Hadamard's inequality bounds the integrand by a product of one-dimensional moments; bounds uniform in θ give $C^{\mathrm{Rice}}_{k,d}$. Section 4.5 assembles the theorem.

1.3 Related work

The Kostlan–Shub–Smale field was introduced by Shub and Smale [16] and Kostlan [10] for the roots of random polynomial systems; its covariance $\langle s,t\rangle^k$ is that of the noise in the spiked tensor model. The Kac–Rice framework for the extrema of smooth Gaussian fields is that of Azaïs and Wschebor [5]. The second-order term of the tail of the maximum, the object of Theorem 1.1, goes back to Azaïs and Wschebor [3] and Azaïs and Wschebor [4], and to Taylor et al. [20] by the Euler characteristic route. The conditional law of the Hessian at a critical point, a shifted GOE, is Lemma 3.2(b) of Auffinger et al. [2], whose Theorems 2.1 and 2.2 are the exact Kac–Rice counts (Proposition 2.3 and Remark 5.3 below) and whose Theorems 2.5, 2.8 and 2.17 give their large-d asymptotics above and below the spectral edge; Theorem 5.1 is the finite-(k,d) counterpart of their Theorems 2.5 and 2.8 above the edge, with the same conditional law as input and the Mehta–Fyodorov evaluation of the expected absolute determinant [14, 9] in place of the large deviations. The connection with spherical spin glasses runs through the p-spin Hamiltonian, whose landscape is studied by Fyodorov [9] and Auffinger et al. [2], and for the spiked model by Ben Arous et al.

[7]. Sharpening the Kac–Rice integral uses the orthogonal-polynomial expansion of the GOE eigenvalue density [14] and Szegő's bounds on the zeros of Hermite polynomials [19]. On the statistical side, $\sup X$ is the injective norm of the noise tensor: in the normalization of Perry et al. [15] its limit is their μ_k, and the detection threshold for the spherical prior sits at $\lambda \asymp \sqrt{d \log k}$ in our variables. The statistical questions attached to the same field are treated in a separate paper and are not used here. The Bayes-optimal side of the same model, the asymptotic mutual information and minimum mean square error of Lesieur et al. [12], is the natural target of a finite-(k, d) bracket for the spiked field; the present paper does not go there, since it never uses the spike.

2 Exact and asymptotically sharp first-order bounds

This section gives the Kac–Rice integral δ_{exact} of (1.1c) in closed form and two explicit relaxations, tightest first: the exact form δ_{exact} of Theorem 2.4, its relaxation δ_{IMF} of Theorem 2.7, and the single-monomial δ_{SMF} of Theorem 2.10. They are nested: $\delta_{\text{exact}} \le \delta_{\text{IMF}}$ on $[u_{\text{IMF}}, \infty)$ (Theorem 2.4) and $\delta_{\text{IMF}} \le \delta_{\text{SMF}}$ on $[u_{\text{SMF}}, \infty)$ (Proposition 2.12). Throughout, δ_{exact} is an equality for $\mathbb{E}N^{\text{cp}}_{[u,\infty)}$ and an upper bound for $\mathbb{P}\{\sup X > u\}$ by (1.1d).

2.1 Mehta–Fyodorov representation (MF)

The Mehta–Fyodorov representation uses two identities: Fyodorov's formula (Lemma 2.1), which writes the expected absolute characteristic polynomial of a $(d-1)$-GOE at level ν through the one-point eigenvalue density of a one-size-larger d-GOE, and Mehta's Hermite expansion of that density (Lemma 2.2); together they give an exact closed algebraic form of the Kac–Rice integrand.

Lemma 2.1 (Fyodorov's formula [9])**.** *For every $\nu \in \mathbb{R}$,*

$$A(\nu) \;=\; \mathbb{E}\big|\det(G - \nu I_{d-1})\big| \;=\; \frac{2^{\frac{3}{2}}\,\Gamma((d+2)/2)}{d}\, Q_d(\nu)\,, \tag{2.1}$$

where $Q_d(\nu) := e^{\nu^2/2} q_d(\nu)$ and q_d is the eigenvalue intensity (one-point correlation function) of a $d \times d$ GOE matrix: $q_d(\nu)\,\mathrm{d}\nu$ is the expected number of eigenvalues in $\mathrm{d}\nu$, so that $\int_{\mathbb{R}} q_d(\nu)\,\mathrm{d}\nu = d$.

The identity expresses the expected absolute characteristic polynomial of the $(d-1) \times (d-1)$ GOE through the one-point function of the one-size-larger $d \times d$ GOE; see Fyodorov [9].

Lemma 2.2 (Mehta expansion [14, p. 221])**.** *With $c_j := (2^j j! \sqrt{\pi})^{-1/2}$ and H_j the j-th physicist Hermite polynomial,*

$$Q_d(\nu) \;=\; \frac{1}{2}\sqrt{\frac{d}{2}}\, c_{d-1} c_d\, H_{d-1}(\nu)\big[\mu_d - 2\mathcal{I}^c_d(\nu)\big] \;+\; \mathbf{1}\{d \text{ odd}\}\,\frac{H_{d-1}(\nu)}{\mu_{d-1}} + e^{-\nu^2/2}\sum_{j=0}^{d-1} c_j^2\, H_j^2(\nu)\,, \tag{2.2}$$

where

$$\mu_m := \int_{\mathbb{R}} H_m(y)\, e^{-y^2/2}\,\mathrm{d}y\,, \qquad \mathcal{I}^c_d(\nu) := \int_{\nu}^{\infty} H_d(y)\, e^{-y^2/2}\,\mathrm{d}y\,.$$

By parity, $\mu_m = 0$ whenever m is odd. For d even the indicator term is absent and the bracket keeps both of its parts. For d odd the constant μ_d vanishes, so the bracket reduces to $-2\mathcal{I}^c_d(\nu)$ and the term $H_{d-1}(\nu)/\mu_{d-1}$ is present; the subtractive part $-2\mathcal{I}^c_d(\nu)H_{d-1}(\nu)$ survives in both parities, and it is the part discarded by the IMF bound.

The combination of Lemmas 2.1 and 2.2 produces an exact integral representation of the Kac–Rice probability bound. Fyodorov's formula (2.1) gives the expected absolute characteristic polynomial as a constant multiple of $Q_d(\rho x)$, and Mehta's expansion (2.2) supplies an explicit formula for Q_d. Substituting

into (1.1c) therefore yields an integrand of the form $Q_d(\rho x)\, e^{-x^2/2}$: a polynomial-and-Hermite quantity weighted by the original Kac–Rice Gaussian, with no further exponential factor entering the integrand. Fyodorov's identity is already stated in the natural variable Q_d, so the eigenvalue intensity $q_d = e^{-\nu^2/2} Q_d$ never appears in the substitution: the Gaussian envelope that would otherwise be carried by q_d is absorbed into the definition of Q_d. This algebraic identity is what the IMF and SMF bounds of Sections 2.3 and 2.4 rest on. We state this in the following proposition.

Proposition 2.3 (Exact MF representation)**.** *For every $u \in \mathbb{R}$,*

$$\mathbb{P}\Big\{ \sup_{\mathbb{S}^{d-1}} X > u \Big\} \;\le\; \delta_{\text{exact}}(u) \;=\; 2(k-1)^{\frac{d-1}{2}} \int_u^\infty Q_d(\rho x)\, e^{-x^2/2}\, \mathrm{d}x\,. \tag{2.3}$$

Proof of Proposition 2.3. • Starting from (1.1c), Lemma 2.1 evaluated at the shifted argument $\nu = \rho x$ gives an exact expression for the expected absolute determinant directly in terms of Q_d:

$$A(\rho x) \;=\; \frac{2^{\frac{3}{2}}\Gamma((d+2)/2)}{d}\, Q_d(\rho x)\,, \tag{2.4}$$

where $Q_d(\nu) = e^{\nu^2/2} q_d(\nu)$ is the unweighted polynomial part of the eigenvalue intensity (one-point correlation function) of a $d \times d$ GOE matrix.
• We multiply this expression by the standard Gaussian density $\varphi(x) = (2\pi)^{-1/2} e^{-x^2/2}$ and integrate over the interval $[u, \infty)$ to match the Kac–Rice integral:

$$\int_u^\infty A(\rho x)\, \varphi(x)\, \mathrm{d}x \;=\; \frac{2^{\frac{3}{2}}\Gamma((d+2)/2)}{d\sqrt{2\pi}} \int_u^\infty Q_d(\rho x)\, e^{-x^2/2}\, \mathrm{d}x\,.$$

We can simplify the constant prefactor using the Gamma function identity $\Gamma((d+2)/2) = (d/2)\Gamma(d/2)$:

$$\frac{2^{\frac{3}{2}}\Gamma((d+2)/2)}{d\sqrt{2\pi}} \;=\; \frac{2\sqrt{2}\cdot(d/2)\Gamma(d/2)}{d\sqrt{2}\sqrt{\pi}} \;=\; \frac{\Gamma(d/2)}{\sqrt{\pi}}\,.$$

Finally, with $C_{k,d} = 2\sqrt{\pi}(k-1)^{\frac{d-1}{2}}/\Gamma(d/2)$ from (1.1c), the $\Gamma(d/2)$ and $\sqrt{\pi}$ terms cancel:

$$C_{k,d}\frac{\Gamma(d/2)}{\sqrt{\pi}} \;=\; \left(\frac{2\sqrt{\pi}(k-1)^{\frac{d-1}{2}}}{\Gamma(d/2)}\right)\frac{\Gamma(d/2)}{\sqrt{\pi}} \;=\; 2(k-1)^{\frac{d-1}{2}}\,.$$

This yields the exact representation (2.3). □

2.2 Strictly exact Mehta–Fyodorov tail bound

Substituting the full Mehta expansion (2.2) into Proposition 2.3 produces the four-piece decomposition

$$\int_u^\infty Q_d(\rho x)\, e^{-x^2/2}\, \mathrm{d}x \;=\; D_1(u) \;+\; D_2(u) \;+\; D_3(u) \;+\; D_4(u)\,, \tag{2.5a}$$

where, with $\nu = \rho x$ and $\Lambda := 2\rho^2 - 1 = 1/(k-1)$,

$$\begin{aligned}
D_1(u) \;&:=\; \tfrac{1}{2}\sqrt{d/2}\, c_{d-1} c_d\, \mu_d\, \mathcal{L}_d(u)\,, && \text{(zero for } d \text{ odd, since } \mu_d = 0),\\
D_2(u) \;&:=\; -\sqrt{d/2}\, c_{d-1} c_d\, \mathcal{C}_d(u)\,, && \text{(cross integral } \mathcal{C}_d \text{ defined in (2.5b))},\\
D_3(u) \;&:=\; \mathbf{1}\{d \text{ odd}\}\, \mu_{d-1}^{-1}\, \mathcal{L}_d(u)\,, && \text{(zero for } d \text{ even)},\\
D_4(u) \;&:=\; T_d^{\text{exact}}(u) && \text{(squared-Hermite tail of (A.1))},
\end{aligned}$$

with $\mathcal{L}_d(u) := \int_u^\infty H_{d-1}(\rho x)\, e^{-x^2/2}\, \mathrm{d}x$ and the cross integral

$$\mathcal{C}_d(u) \;:=\; \int_u^\infty H_{d-1}(\rho x)\, \mathcal{I}_d^c(\rho x)\, e^{-x^2/2}\, \mathrm{d}x\,. \tag{2.5b}$$

Theorem 2.4 (Strictly exact MF tail bound)**.** *For every $u \in \mathbb{R}$,*

$$\mathbb{P}\Big\{ \sup_{\mathbb{S}^{d-1}} X > u \Big\} \;\le\; \delta_{\mathrm{exact}}(u) \;=\; 2(k-1)^{\frac{d-1}{2}} \big[D_1(u) + D_2(u) + D_3(u) + D_4(u) \big] , \tag{2.6a}$$

where each D_i is given in fully closed form by:

(D_1, D_3) *the linear Hermite tail $\mathcal{L}_d(u)$ admits the closed form*

$$\mathcal{L}_d(u) \;=\; \frac{1}{\rho}\, J_{d-1}^{1/\rho^2}(\rho u) , \tag{2.6b}$$

with J_m^β given by Lemma B.4;

(D_4) *the squared Hermite tail $T_d^{\mathrm{exact}}(u) = \rho^{-1} \sum_{j=0}^{d-1} c_j^2\, K_j^\beta(\rho u)$ of (A.1), with $\beta = (3k-2)/k$ and K_j^β given by Corollary B.5;*

(D_2) *the cross integral $\mathcal{C}_d(u)$ admits the two-piece closed form*

$$\begin{aligned} \mathcal{C}_d(u) \;=\; & \frac{1}{\rho} \sum_{\ell=0}^{\lfloor (d-1)/2 \rfloor} \frac{2^{\ell+1}\,(d-1)!!}{(d-2\ell-1)!!} \sum_{p=0}^{d-2\ell-1} 2^p\, p! \binom{d-1}{p} \binom{d-2\ell-1}{p} J_{2d-2\ell-2-2p}^\beta(\rho u) \\ & +\; \mathbf{1}\{d \,\text{even}\}\, 2^{d/2}\,(d-1)!!\, \mathcal{F}_d(u) , \end{aligned} \tag{2.6c}$$

where the Fubini remainder $\mathcal{F}_d(u)$ (only present for d even) reads

$$\mathcal{F}_d(u) \;=\; \sqrt{2\pi}\, \bar{\Phi}(\rho u)\, \frac{J_{d-1}^{1/\rho^2}(\rho u)}{\rho} \;-\; 2\rho \sum_{\ell=0}^{(d-2)/2} (2\Lambda)^\ell\, \frac{(d-2)!!}{(d-2\ell-2)!!}\, J_{d-2\ell-2}^\beta(\rho u) . \tag{2.6d}$$

Moreover, with u_{IMF}, α_d, Φ_d and δ_{IMF} as in Proposition 2.6 and Theorem 2.7,

$$\begin{aligned} & \delta_{\mathrm{exact}}(u) \;<\; \delta_{\mathrm{IMF}}^\star(u) \;<\; \delta_{\mathrm{IMF}}(u) \qquad \text{for every } u \ge u_{\mathrm{IMF}}, \\ & \text{where} \quad \delta_{\mathrm{IMF}}^\star(u) := 2(k-1)^{\frac{d-1}{2}} \big[\alpha_d \Phi_d(\rho, u) e^{-u^2/2} + D_4(u) \big]. \end{aligned}$$

Since $\beta > 2$, the sums of Lemma B.4 alternate. The quantity $2(k-1)^{(d-1)/2} \sum_{i=1}^4 D_i(u)$ is the closed-form evaluation of the Kac–Rice integral $2(k-1)^{(d-1)/2} \int_u^\infty Q_d(\rho x) e^{-x^2/2}\, \mathrm{d}x$ of Proposition 2.3, that is of $\delta_{\mathrm{exact}}(u) = \mathbb{E} N_{[u,\infty)}^{\mathrm{cp}}$ in (1.1c); the inequality (2.6a) for $\mathbb{P}\{\sup X > u\}$ is (1.1d). Both readings are used below: the second in Sections 3 to 4.5, where the upper bound is matched by a lower one, and the first in Section 5, where the count is the annealed complexity. For k even, $X(-s) = X(s)$ and the critical points come in antipodal pairs, so Proposition 4.10 halves every bound of this section on $\mathbb{P}\{\sup X > u\}$; the bounds on $\mathbb{E} N_{[u,\infty)}^{\mathrm{cp}}$ are unchanged.

Proof. Deferred to Appendix A. □

Figure 1 compares the three bounds of Table 1 with the baseline at $(k, d) = (3, 5)$ and $(4, 6)$; the figure is produced by the notebook of `github.com/ydecastro/TENSOR-KSS`.

2.3 Improved Mehta–Fyodorov bound (IMF)

Inspecting the Mehta expansion (2.2), one sees that the coefficient of the Hermite polynomial $H_{d-1}(\nu)$ is the bracket $[\mu_d - 2\mathcal{I}_d^c(\nu)]$, where $\mathcal{I}_d^c(\nu) := \int_\nu^\infty H_d(y) e^{-y^2/2}\, \mathrm{d}y$ is a Hermite tail integral. If a regime can be identified in which $\mathcal{I}_d^c(\nu)$ is *strictly positive*, the subtractive contribution $-2\mathcal{I}_d^c(\nu) H_{d-1}(\nu)$ to Q_d is strictly negative (since $H_{d-1}(\nu) > 0$ in the same regime), and discarding it yields a strict upper bound on the Kac–Rice integral.

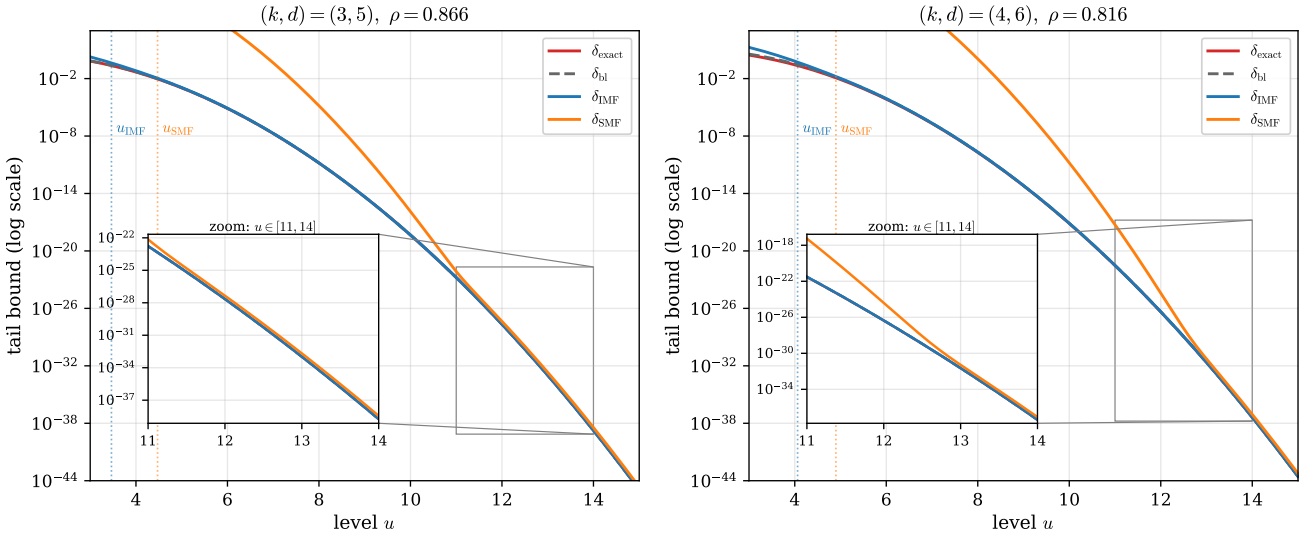


Figure 1: The bounds δ_{exact} (Theorem 2.4), δ_{IMF} (Theorem 2.7) and δ_{SMF} (Theorem 2.10), and the asymptotic baseline δ_{bl} of Table 1, which is not a bound, on logarithmic scale against the level u, for $(k,d)=(3,5)$ (left) and $(4,6)$ (right). The dotted vertical lines mark the validity thresholds u_{IMF} and u_{SMF}; each curve is drawn on the whole window. At this scale δ_{IMF} is indistinguishable from δ_{exact}, the ratio being 1.014 at $u=12$ for $(3,5)$ and 1.032 for $(4,6)$; δ_{bl} tracks δ_{exact} from above, the ratio $\delta_{\text{exact}}/\delta_{\text{bl}}$ being 0.993 at $u=12$ for $(3,5)$ and 0.976 for $(4,6)$; δ_{SMF} sits above δ_{exact} by the factor 2 of its main term at $(3,5)$, a ratio of 2.02 at $u=12$, and by its remainder term $u^{2d-3}e^{-3u^2/4}$ at $(4,6)$, a ratio of about 91 at $u=12$. The inset zooms on $u\in[11,14]$.

Pinpointing this regime reduces to locating the largest root of H_{d-1}: beyond it, $H_{d-1}(\nu)$ is positive, and by a root-interlacing argument $\mathcal{I}_d^c(\nu)$ is too. Hence the threshold is set by the largest root of H_{d-1}. Szegő's classical estimate [19, Theorem 6.32] locates this largest root strictly below $\sqrt{2d-1}$, so that the relevant regime is $\nu\geq\sqrt{2d-1}$, which translates into the threshold $\rho u\geq\sqrt{2d-1}$ in the Kac–Rice integral.

Lemma 2.5 (Positivity of the Hermite tail integral)**.** *Let $d\geq 3$ and let $x_1^{(m)}$ denote the largest root of the (physicist) Hermite polynomial H_m. For every $\nu\geq x_1^{(d-1)}$, the tail integral $\mathcal{I}_d^c(\nu)$ is strictly positive.*

Proof of Lemma 2.5. • We apply the explicit finite series expansion from Lemma B.3 (specifically Equation (B.2)) with degree $m=d$. This yields:

$$\mathcal{I}_d^c(\nu)=e^{-\nu^2/2}\sum_{\ell=0}^{\lfloor(d-1)/2\rfloor}2^{\ell+1}\frac{(d-1)!!}{(d-2\ell-1)!!}H_{d-2\ell-1}(\nu)+R_d(\nu)\,,$$

where the remainder term guarantees $R_d(\nu)\geq 0$.

• The roots of consecutive orthogonal polynomials interlace [19, Theorem 3.3.2]; in particular, the largest root $x_1^{(m)}$ of H_m is strictly increasing in the degree m. Therefore, for every $\ell\geq 1$, we have the strict inequality:

$$x_1^{(d-2\ell-1)}<x_1^{(d-1)}\,.$$

When we evaluate the sum at any $\nu\geq x_1^{(d-1)}$, the leading polynomial ($\ell=0$) satisfies $H_{d-1}(\nu)\geq 0$. For all subsequent lower-degree polynomials ($\ell\geq 1$; there is at least one since $d\geq 3$), since $\nu\geq x_1^{(d-1)}>x_1^{(d-2\ell-1)}$, they evaluate to strictly positive values: $H_{d-2\ell-1}(\nu)>0$. Because all the constant combinatorial coefficients in the sum are strictly positive and $R_d(\nu)\geq 0$, the entire expression evaluates to a strictly positive number: $\mathcal{I}_d^c(\nu)>0$. □

With the positivity of $\mathcal{I}_d^c$ secured on $\{\nu \geq x_1^{(d-1)}\}$, the subtractive contribution in the Mehta expansion may then be discarded. The two remaining pieces (the linear Hermite tail $\int_u^\infty H_{d-1}(\rho x)e^{-x^2/2}\,\mathrm{d}x$ and the squared Hermite tail $\sum_j c_j^2 \int_u^\infty H_j^2(\rho x)e^{-(1+\rho^2)x^2/2}\,\mathrm{d}x$) decay at different Gaussian rates, $e^{-u^2/2}$ and $e^{-(1+\rho^2)u^2/2}$ respectively. This two-scale structure is preserved in the proposition below; the SMF bound of the next subsection collapses it to a single dominant scale. Analytically, we evaluate the linear tail by iterating the three-term Hermite recurrence, and the squared tail by substituting the envelope $H_j(\rho x) \leq (2\rho x)^j$ of Lemma 2.8 into each squared term and invoking Lemma B.1.

Proposition 2.6 (Explicit IMF upper bound on Q_d)**.** *For every $u > 0$ such that $\rho u \geq \sqrt{2d-1}$,*

$$\int_u^\infty Q_d(\rho x)\, e^{-x^2/2}\,\mathrm{d}x \;\leq\; \alpha_d\, \Phi_d(\rho, u)\, e^{-u^2/2} \;+\; \Psi_d(\rho, u)\, e^{-(1+\rho^2)u^2/2}\,, \tag{2.7a}$$

where α_d is the dominant Mehta coefficient of Lemma 2.9, and the explicit polynomial–rational functions are given, with $\Lambda = 2\rho^2 - 1$, by

$$\Phi_d(\rho, u) \;:=\; \sum_{\ell=0}^{\lfloor (d-2)/2 \rfloor} 2\rho\,(2\Lambda)^\ell\, \frac{(d-2)!!}{(d-2\ell-2)!!}\,(2\rho u)^{d-2\ell-2} \;+\; \frac{\mathbf{1}\{d \text{ odd}\}}{u}\,(2\Lambda)^{\frac{d-1}{2}}\,(d-2)!!\,, \tag{2.7b}$$

$$\Psi_d(\rho, u) \;:=\; \frac{c_0^2}{(1+\rho^2)\,u} \;+\; \sum_{j=1}^{d-1} c_j^2\, \frac{(2\rho)^{2j}\, u^{2j-1}}{1+\rho^2-(2j-1)/u^2}\,. \tag{2.7c}$$

Proof of Proposition 2.6. • By Szegő's bound, the largest root of H_{d-1} strictly satisfies $x_1^{(d-1)} < \sqrt{2d-1}$. The proposition assumes $\rho u \geq \sqrt{2d-1}$, ensuring that $\rho x > x_1^{(d-1)}$ for all integration variables $x \geq u$. By Lemma 2.5, this guarantees $\mathcal{I}_d^c(\rho x) > 0$ over the entire integration domain. Returning to the exact Mehta expansion (2.2), the contribution of $-2\mathcal{I}_d^c(\rho x)H_{d-1}(\rho x)$ is therefore strictly negative. Discarding it yields a strict upper bound:

$$\int_u^\infty Q_d(\rho x)e^{-x^2/2}\,\mathrm{d}x \;\leq\; \alpha_d\,\mathcal{L}_d(u) \;+\; \mathcal{S}_d(u)\,, \tag{2.8}$$

where α_d is the dominant Mehta coefficient given by (2.10c), and the components are defined as:

$$\mathcal{L}_d(u) := \int_u^\infty H_{d-1}(\rho x)e^{-x^2/2}\,\mathrm{d}x\,,$$
$$\mathcal{S}_d(u) := \sum_{j=0}^{d-1} c_j^2 \int_u^\infty H_j^2(\rho x)e^{-(1+\rho^2)x^2/2}\,\mathrm{d}x\,.$$

• We derive a generalized recurrence for $J_m := \int_u^\infty H_m(\rho x)e^{-x^2/2}\,\mathrm{d}x$. Applying the identity $H_m(\rho x) = 2\rho x H_{m-1}(\rho x) - 2(m-1)H_{m-2}(\rho x)$ and evaluating the first term via integration by parts, we obtain:

$$J_m = 2\rho\, H_{m-1}(\rho u)\, e^{-u^2/2} + 2\Lambda(m-1)\, J_{m-2}\,, \qquad \text{where } \Lambda = 2\rho^2 - 1\,,$$

where we used $\frac{\mathrm{d}}{\mathrm{d}x}[H_{m-1}(\rho x)e^{-x^2/2}] = 2\rho(m-1)H_{m-2}(\rho x)e^{-x^2/2} - xH_{m-1}(\rho x)e^{-x^2/2}$. Unrolling this recurrence for $m = d-1$ produces a finite series of boundary evaluations. We bound each boundary evaluation using $H_{d-2\ell-2}(\rho u) \leq (2\rho u)^{d-2\ell-2}$ (Lemma 2.8). If $d-1$ is even (i.e., d is odd), the recurrence bottoms out with a Gaussian tail, which we bound using Mills' ratio $\int_u^\infty e^{-x^2/2}\,\mathrm{d}x \leq u^{-1}e^{-u^2/2}$. Factoring out $e^{-u^2/2}$ yields exactly:

$$\mathcal{L}_d(u) \;\leq\; \Phi_d(\rho, u)\, e^{-u^2/2}\,,$$

where Φ_d is the polynomial-rational function defined in (2.7b).

• For the squared component, we substitute the bound $H_j(\rho x) \le (2\rho x)^j$ of Lemma 2.8 directly into the integral:

$$\mathcal{S}_d(u) \;\le\; \sum_{j=0}^{d-1} c_j^2 \int_u^\infty (2\rho x)^{2j} e^{-(1+\rho^2)x^2/2}\,\mathrm{d}x\,.$$

We evaluate each integral using Lemma B.1 with parameters $a = 1+\rho^2$ and $m = 2j$; its closed form applies since $(2j-1)/u^2 < \rho^2 < a$ for $\rho u \ge \sqrt{2d-1}$. For the base case $j = 0$, we use the Mills bound $I_0(u;a) \le (au)^{-1}e^{-au^2/2}$ of (B.1a). Factoring out the shared exponential weight $e^{-(1+\rho^2)u^2/2}$ leaves the rational function $\Psi_d(\rho,u)$ defined in (2.7c), yielding:

$$\mathcal{S}_d(u) \;\le\; \Psi_d(\rho,u)\, e^{-(1+\rho^2)u^2/2}\,.$$

• Substituting the bounds for $\mathcal{L}_d(u)$ and $\mathcal{S}_d(u)$ back into (2.8) completes the proof:

$$\int_u^\infty Q_d(\rho x) e^{-x^2/2}\,\mathrm{d}x \;\le\; \alpha_d\,\Phi_d(\rho,u)\,e^{-u^2/2} \;+\; \Psi_d(\rho,u)\,e^{-(1+\rho^2)u^2/2}\,. \qquad \square$$

Theorem 2.7 (IMF tail bound)**.** *For all $u \ge u_{\mathrm{IMF}} = \sqrt{2d-1}/\rho$,*

$$\mathbb{P}\Big\{\sup_{\mathbb{S}^{d-1}} X > u\Big\} \;\le\; \delta_{\mathrm{IMF}}(u) \;:=\; 2(k-1)^{\frac{d-1}{2}}\Big(\alpha_d\,\Phi_d(\rho,u)\,e^{-u^2/2} + \Psi_d(\rho,u)\,e^{-(1+\rho^2)u^2/2}\Big)\,.$$

Proof of Theorem 2.7. Apply Proposition 2.6 to the right-hand side of (2.3) and substitute the constant prefactor $2(k-1)^{\frac{d-1}{2}}$ from Proposition 2.3. $\square$

Sharpness of the IMF bound. Comparing the leading coefficients in Lemmas 2.1 and 2.9 with $A(\nu) \sim \nu^{d-1}$ (Lemma B.2) gives

$$\alpha_d \;=\; \big(\sqrt{2}\,2^{d-1}\Gamma(d/2)\big)^{-1}; \tag{2.9}$$

hence $\delta_{\mathrm{IMF}}(u) \sim \delta_{\mathrm{bl}}(u) \sim \delta_{\mathrm{exact}}(u)$ as $u \to \infty$, which is the sense in which the bound is asymptotically sharp.

2.4 Simplified Mehta–Fyodorov bound (SMF)

The SMF bound applies a second relaxation beyond IMF: the envelopes $H_{d-1}(\nu) < (2\nu)^{d-1}$ and $(\nu^2+1)^{d-1}$ collapse Φ_d and the squared Hermite sum to a single monomial main term $u^{d-2}e^{-u^2/2}$ plus a faster $e^{-3u^2/4}$ remainder, at the cost of a uniform factor 2 in the leading constant. Beyond the level $u_d^\star$ the remainder is dominated and the bound collapses to a single monomial (Corollary 2.11).

Lemma 2.8 (Hermite envelope)**.** *For every $m \ge 2$ and every $x > x_1^{(m)}$, $0 < H_m(x) < (2x)^m$.*

Proof of Lemma 2.8. Write $H_m(x) = 2^m \prod_{j=1}^m (x - x_j^{(m)})$. For $x > x_1^{(m)}$, every factor is strictly positive, so $H_m(x) > 0$. For the upper bound, the inequality $H_m(x) < (2x)^m$ is equivalent to $\prod_{j=1}^m (1 - x_j^{(m)}/x) < 1$. Taking logarithms,

$$\sum_{j=1}^m \log(1 - x_j^{(m)}/x) \;<\; -\sum_{j=1}^m x_j^{(m)}/x \;=\; 0\,,$$

where the strict inequality uses $\log(1-t) < -t$ for $t \ne 0$ and the equality uses the vanishing sum of Hermite roots, which follows from the absence of the x^{m-1} term in $H_m(x)$. $\square$

Lemma 2.9 (Non-asymptotic decomposition of Q_d). *With c_j as in Lemma 2.2, $S_d := \sum_{j=0}^{d-1} c_j^2\, 2^j \big[\frac{(2j)!}{j!}\big]^2$, $\widetilde{B}_d := \frac{(2d)!\,2^{d+1}}{d!}$ and*

$$\beta_d := S_d + \sqrt{d/2}\, c_{d-1} c_d \, \frac{(2d-2)!}{(d-1)!}\, 2^{d-1}\, \widetilde{B}_d\,, \tag{2.10a}$$

$$Q_d(\nu) \;=\; \alpha_d\, H_{d-1}(\nu) \;+\; \mathcal{R}_d(\nu)\,, \tag{2.10b}$$

with the dominant coefficient given by

$$\alpha_d \;=\; \begin{cases} \frac{1}{2}\sqrt{d/2}\, c_{d-1} c_d\, \mu_d & \text{if } d \text{ even}, \\ 1/\mu_{d-1} & \text{if } d \text{ odd}, \end{cases} \tag{2.10c}$$

and the remainder bound, for every $\nu \geq 0$,

$$|\mathcal{R}_d(\nu)| \;\leq\; \beta_d\, (\nu^2+1)^{d-1}\, e^{-\nu^2/2}\,. \tag{2.10d}$$

Proof. Deferred to Appendix A. □

The decomposition (2.10b) translates into a non-asymptotic upper bound on the Kac–Rice integrand:

$$A(\nu) \;\leq\; \alpha_d\, \frac{2^{\frac{3}{2}}\Gamma((d+2)/2)}{d}\, (2\nu)^{d-1} \;+\; \frac{2^{\frac{3}{2}}\Gamma((d+2)/2)}{d}\, \beta_d\, (\nu^2+1)^{d-1}\, e^{-\nu^2/2}\,.$$

Inserting this inequality into the Kac–Rice integral (1.1c) and applying the Gaussian-type tail integrals of Lemma B.1 (Appendix B.1) yields the SMF bound.

Theorem 2.10 (SMF tail bound). *Let $u_{\mathrm{SMF}} = 2\sqrt{d}$. For all $u \geq u_{\mathrm{SMF}}$,*

$$\mathbb{P}\Big\{ \sup_{\mathbb{S}^{d-1}} X > u \Big\} \;\leq\; \delta_{\mathrm{SMF}}(u) \;:=\; 4\,\alpha_d\, (2k)^{\frac{d-1}{2}}\, u^{d-2}\, e^{-u^2/2} \;+\; 2^d\, \beta_d\, (k-1)^{\frac{d-1}{2}}\, u^{2d-3}\, e^{-3u^2/4}\,. \tag{2.11}$$

Proof of Theorem 2.10. • By Proposition 2.3, the excursion probability is bounded by the integral of the Mehta function:

$$\mathbb{P}\Big\{ \sup_{\mathbb{S}^{d-1}} X > u \Big\} \;\leq\; 2(k-1)^{\frac{d-1}{2}} \int_u^\infty Q_d(\rho x)\, e^{-x^2/2}\, \mathrm{d}x\,.$$

Using the non-asymptotic decomposition from Lemma 2.9, we write $Q_d = \alpha_d H_{d-1} + \mathcal{R}_d$. This splits the corresponding integral into two parts: $I_{\mathrm{main}} + I_{\mathrm{rem}}$.

• The main term integral evaluates the dominant Hermite polynomial. By hypothesis, $u \geq u_{\mathrm{SMF}} = 2\sqrt{d}$ and $\rho^2 \geq 1/2$ (since $k \geq 3$). This ensures that $\rho u \geq \sqrt{2d} > \sqrt{2d-1}$, meaning $\rho x > x_1^{(d-1)}$ for all integration variables $x \geq u$. Applying the polynomial envelope from Lemma 2.8, we have $0 < H_{d-1}(\rho x) < (2\rho x)^{d-1}$. Substituting this strictly positive bound into the main integral gives:

$$I_{\mathrm{main}} \;<\; \alpha_d (2\rho)^{d-1} \int_u^\infty x^{d-1} e^{-x^2/2}\, \mathrm{d}x\,.$$

We evaluate this using Lemma B.1 with $a = 1$ and $m = d-1$. The required tail hypothesis $(d-2)/u^2 \leq 1/2$ holds safely for $u \geq 2\sqrt{d}$, yielding:

$$I_{\mathrm{main}} \;\leq\; 2\alpha_d (2\rho)^{d-1} u^{d-2} e^{-u^2/2}\,.$$

• Using the remainder envelope (2.10d), the absolute value of the remainder integral is bounded by:

$$|I_{\mathrm{rem}}| \;\leq\; \beta_d \int_u^\infty (\rho^2 x^2 + 1)^{d-1} e^{-(1+\rho^2)x^2/2}\, \mathrm{d}x\,.$$

For the polynomial factor, we factor $\rho^2 x^2$ out exactly:

$$(\rho^2 x^2 + 1)^{d-1} \;=\; (\rho^2 x^2)^{d-1} \left(1 + 1/(\rho^2 x^2)\right)^{d-1} \;=\; \rho^{2(d-1)}\, x^{2(d-1)} \left(1 + 1/(\rho^2 x^2)\right)^{d-1}.$$

Since $\rho^2 \geq 1/2$ (because $k \geq 3$) and $x \geq u \geq 2\sqrt{d}$, we have $\rho^2 x^2 \geq 2d \geq 1$, so the correction factor satisfies $(1 + 1/(\rho^2 x^2))^{d-1} \leq 2^{d-1}$. Combined with $\rho \leq 1$, which gives $\rho^{2(d-1)} \leq 1$, this yields $(\rho^2 x^2 + 1)^{d-1} \leq 2^{d-1}\, x^{2(d-1)}$. For the exponential factor, since $k \geq 3$, we know $1 + \rho^2 \geq 3/2$. Substituting these simplifications gives:

$$|I_{\mathrm{rem}}| \;\leq\; 2^{d-1} \beta_d \int_u^\infty x^{2(d-1)} e^{-3x^2/4}\, \mathrm{d}x\,.$$

Applying Lemma B.1 with $a = 3/2$ and $m = 2(d-1)$ (its second particular case, valid for $u \geq 2\sqrt{d}$), we obtain:

$$|I_{\mathrm{rem}}| \;\leq\; 2^{d-1} \beta_d\, u^{2d-3} e^{-3u^2/4}\,.$$

• We combine the bounded integrals and multiply by the global prefactor $2(k-1)^{\frac{d-1}{2}}$. For the main term, the identity $(2\rho)^{d-1}(k-1)^{\frac{d-1}{2}} = (2k)^{\frac{d-1}{2}}$ consolidates the constants. Summing the two appropriately scaled components yields the final bound:

$$\mathbb{P}\Big\{ \sup_{\mathbb{S}^{d-1}} X > u \Big\} \;\leq\; 4\alpha_d (2k)^{\frac{d-1}{2}} u^{d-2} e^{-u^2/2} \;+\; 2^d \beta_d (k-1)^{\frac{d-1}{2}} u^{2d-3} e^{-3u^2/4}\,. \qquad \square$$

The two-term structure of $\delta_{\mathrm{SMF}}(u)$ reflects the two exponential scales inherited from the Mehta expansion: the $e^{-u^2/2}$ scale of the linear Hermite tail, which matches the optimal Kac–Rice decay rate, and the $e^{-3u^2/4}$ scale of the squared Hermite tail relaxed through the $(\nu^2+1)^{d-1}$ envelope. Since the latter is strictly faster, the remainder is asymptotically negligible and the bound collapses to its main term for sufficiently large u. The following corollary pinpoints the explicit threshold $u_d^\star \geq u_{\mathrm{SMF}}$ beyond which this collapse occurs, at the mild price of a factor of 2 in the leading prefactor.

Corollary 2.11 (Asymptotic single-term form)**.** *There exists an explicit threshold $u_d^\star \geq u_{\mathrm{SMF}}$, depending only on (k, d), such that for every $u \geq u_d^\star$ the remainder term in (2.11) is dominated by the main term, in which case*

$$\mathbb{P}\Big\{ \sup_{\mathbb{S}^{d-1}} X > u \Big\} \;\leq\; 8\, \alpha_d\, (2k)^{\frac{d-1}{2}}\, u^{d-2}\, e^{-u^2/2}\,.$$

Explicitly, $u_d^\star$ is the smallest $u \geq 2\sqrt{d}$ satisfying $2^{d-2} \beta_d\, u^{d-1}\, e^{-u^2/4} \leq \alpha_d (2\rho)^{d-1}$.

Proof of Corollary 2.11. Since $\big((k-1)/(2k)\big)^{(d-1)/2} = (2\rho)^{-(d-1)}$, the ratio of the remainder term of (2.11) to its main term is $R(u) = 2^{d-2}\beta_d\, u^{d-1} e^{-u^2/4} / \big(\alpha_d (2\rho)^{d-1}\big)$, which is decreasing on $[\sqrt{2(d-1)}, \infty) \supset [2\sqrt{d}, \infty)$. Hence $R(u) \leq 1$ for every $u \geq u_d^\star$, and there the bound of Theorem 2.10 is at most twice its main term, $8\alpha_d (2k)^{(d-1)/2} u^{d-2} e^{-u^2/2}$. $\square$

Proposition 2.12 (Domination of the SMF bound by the IMF bound)**.** *For every $k \geq 3$ and $d \geq 3$,*

$$\delta_{\mathrm{IMF}}(u) \;\leq\; \delta_{\mathrm{SMF}}(u) \qquad \text{for every } u \geq u_{\mathrm{SMF}} = 2\sqrt{d}\,. \tag{2.12}$$

The proof uses the following envelope of Φ_d.

Lemma 2.13 (Geometric envelope on Φ_d)**.** *For every $k \geq 3$, $d \geq 3$, and every $u \geq u_{\mathrm{SMF}} = 2\sqrt{d}$,*

$$\Phi_d(\rho, u) \;\leq\; 2\, (2\rho)^{d-1}\, u^{d-2}\,. \tag{2.13}$$

Proof. Since $\Lambda = 2\rho^2 - 1 = 1/(k-1)$ (as in (2.7b)), we have $2\Lambda = 2/(k-1)$, and we set

$$q \;:=\; \frac{2\Lambda}{(2\rho u)^2} \;=\; \frac{2}{(k-1)\,(2\rho u)^2}\,.$$

On $\{u \ge 2\sqrt{d}\}$, $\rho^2 \ge 1/2$ for $k \ge 3$, hence $(2\rho u)^2 \ge 4\rho^2 \cdot 4d \ge 8d$, and therefore

$$(d-2)\,q \;\le\; \frac{2(d-2)}{8d\,(k-1)} \;=\; \frac{d-2}{4d\,(k-1)} \;\le\; \frac{1}{4(k-1)} \;\le\; \frac{1}{8}\,.$$

The leading term of Φ_d (the $\ell = 0$ summand of (2.7b)) equals $2\rho\,(2\rho u)^{d-2} = (2\rho)^{d-1}u^{d-2}$. The ℓ-th summand for $\ell \ge 1$, divided by the leading term, equals $(d-2)(d-4)\cdots(d-2\ell)\,q^\ell$, a product of ℓ factors each at most $(d-2)q \le 1/8$; hence

$$\sum_{\ell \ge 1} \frac{\ell\text{-th summand}}{\text{leading}} \;\le\; \sum_{\ell \ge 1} (1/8)^\ell \;=\; \frac{1}{7}\,.$$

Similarly, for d odd, the trailing term $\mathbf{1}\{d\,\text{odd}\}\,u^{-1}\,(2\Lambda)^{(d-1)/2}\,(d-2)!!$ divided by the leading is $q^{(d-1)/2}\,(d-2)!! = \prod_{j=1}^{(d-1)/2}(d-2j)\,q \le (1/8)^{(d-1)/2} \le 1/8$ (the last bound uses $(d-1)/2 \ge 1$ for $d \ge 3$). Summing,

$$\Phi_d(\rho,u)/\big((2\rho)^{d-1}u^{d-2}\big) \;\le\; 1 + \tfrac{1}{7} + \tfrac{1}{8} \;=\; \tfrac{71}{56} \;\le\; 2\,,$$

which is (2.13). □

Proof of Proposition 2.12. We bound the IMF main and residual terms separately on $[u_{\mathrm{SMF}}, \infty)$.

• *Main term.* On the interval $[u_{\mathrm{SMF}}, \infty)$, Lemma 2.13 gives $\Phi_d(\rho,u) \le 2(2\rho)^{d-1}u^{d-2}$. Multiply by $2(k-1)^{(d-1)/2}\alpha_d e^{-u^2/2}$ and use $\rho^{d-1}(k-1)^{(d-1)/2} = (k/2)^{(d-1)/2}$:

$$\underbrace{2(k-1)^{\frac{d-1}{2}}\,\alpha_d\,\Phi_d(\rho,u)\,e^{-u^2/2}}_{\text{IMF main}} \;\le\; 4\,\alpha_d\,(2\rho)^{d-1}\,(k-1)^{\frac{d-1}{2}}\,u^{d-2}\,e^{-u^2/2}$$

$$= \;4\,\alpha_d\,(2k)^{\frac{d-1}{2}}\,u^{d-2}\,e^{-u^2/2} \;=\; \underbrace{\delta_{\mathrm{SMF}}^{\mathrm{main}}(u)}_{\text{first term of (2.11)}}\,.$$

• *Residual.* On $[u_{\mathrm{SMF}}, \infty)$, $u^2 \ge 4d$ so $(2j-1)/u^2 \le (2d-3)/(4d) \le 1/2$ for every $j \le d-1$; combined with $1+\rho^2 \ge 3/2$ (since $k \ge 3$), the rational factor in (2.7c) satisfies

$$\frac{1}{1+\rho^2-(2j-1)/u^2} \;\le\; \frac{1}{1+\rho^2-1/2} \;\le\; 1\,.$$

Hence

$$\Psi_d(\rho,u) \;\le\; \frac{c_0^2}{(1+\rho^2)\,u} + \sum_{j=1}^{d-1} c_j^2\,(2\rho)^{2j}\,u^{2j-1} \;\le\; u^{2d-3}\sum_{j=0}^{d-1} c_j^2\,(2\rho)^{2j}\,,$$

using $u^{2j-1} \le u^{2d-3}$ for $j \le d-1$ and $u \ge 1$, the trivial bound $1/[(1+\rho^2)u] \le u^{2d-3}$, and $1/(1+\rho^2) \le 1 \le (2\rho)^0 = 1$ to consolidate the $j=0$ term. Comparing the per-j Szegő-style coefficient $c_j^2(2\rho)^{2j} = c_j^2\,4^j\rho^{2j}$ to the entry $c_j^2\,2^j\,((2j)!/j!)^2$ of S_d in the definition (2.10a) of β_d: since $(2j)!/j! = (j+1)(j+2)\cdots(2j) \ge 2^j$ for $j \ge 1$, $((2j)!/j!)^2 \ge 4^j$, and using $\rho \le 1$, $(2\rho)^{2j} = 4^j\rho^{2j} \le 4^j \le 2^j \cdot 4^j \le 2^j((2j)!/j!)^2$, hence $c_j^2(2\rho)^{2j} \le c_j^2\,2^j((2j)!/j!)^2$ termwise (for $j=0$ the comparison reads $c_0^2 \le c_0^2$ with equality), giving $\sum_j c_j^2(2\rho)^{2j} \le S_d \le \beta_d$. Therefore

$$\underbrace{2(k-1)^{\frac{d-1}{2}}\,\Psi_d(\rho,u)\,e^{-(1+\rho^2)u^2/2}}_{\text{IMF residual}} \;\le\; 2(k-1)^{\frac{d-1}{2}}\,\beta_d\,u^{2d-3}\,e^{-(1+\rho^2)u^2/2}\,.$$

Comparing to $\delta_{\mathrm{SMF}}^{\mathrm{rem}}(u) := 2^d\beta_d(k-1)^{(d-1)/2}u^{2d-3}e^{-3u^2/4}$, the ratio of exponentials is $e^{-(1+\rho^2)u^2/2+3u^2/4} = e^{-\Lambda u^2/4}$, which lies in $(0,1]$ for $\Lambda = 2\rho^2-1 = 1/(k-1) > 0$ (i.e., $k \ge 3$). The ratio of polynomial prefactors is $2/2^d = 2^{1-d} \le 1$ for $d \ge 1$. Hence the IMF residual is bounded by $\delta_{\mathrm{SMF}}^{\mathrm{rem}}$ on $[u_{\mathrm{SMF}}, \infty)$.

Adding the two pieces, $\delta_{\mathrm{IMF}}(u) \le \delta_{\mathrm{SMF}}^{\mathrm{main}}(u) + \delta_{\mathrm{SMF}}^{\mathrm{rem}}(u) = \delta_{\mathrm{SMF}}(u)$ on $[u_{\mathrm{SMF}}, \infty)$, which is (2.12). □

3 The gap as a marked Rice integral, and the layer decomposition

Lemma 3.1. *For $u > 0$, $\mathbb{P}\{\sup X > u\} \le \delta_{\mathrm{exact}}(u)$ and, for k odd,*

$$\delta_{\mathrm{exact}}(u) - \mathbb{P}\{\sup X > u\} = \mathbb{E}\,\#\big\{t :\ \nabla X(t) = 0,\ X(t) \ge u,\ \exists s :\ X(s) > X(t)\big\}. \tag{3.1}$$

Proof. The first claim is (1.1d). For k odd, $\mathrm{Var}(X(s) - X(t)) = 2(1 - \langle s,t\rangle^k)$ vanishes only for $s = t$, so the maximum is almost surely attained at a single point t_*, a consequence of Tsirelson's theorem [13, Theorem 3]. A critical point above u is either t_*, exactly on $\{\sup X \ge u\}$, or beaten; hence $N^{\mathrm{cp}}_{[u,\infty)} = \mathbf{1}\{\sup X \ge u\} + \#\{\text{beaten critical points above } u\}$, and $\mathbb{P}\{\sup X = u\} \le \mathbb{E}N^{\mathrm{cp}}_{\{u\}} = 0$ by (1.1c), so $\mathbb{P}\{\sup X \ge u\} = \mathbb{P}\{\sup X > u\}$; both sides of (3.1) are finite by (1.1c). □

Fix $t \in \mathbb{S}^{d-1}$ and write $s = \cos\theta\, t + \sin\theta\, v$ with $\theta \in (0,\pi)$, $v \in \mathbb{S}^{d-2}$ the unit sphere of $t^\perp$, $r = \cos\theta = \langle s,t\rangle$, and $M := \mathbb{S}^{d-1} \setminus \{\pm t\}$. Standard computations give $\mathrm{Cov}(X(s), X(t)) = r^k$, $\mathrm{Cov}(X(s), \partial_a X(t)) = k\, r^{k-1} \sin\theta\, v_a$, $\mathrm{Cov}(X(t), \nabla X(t)) = 0$ and $\mathrm{Cov}(\nabla X(t)) = k I_{d-1}$, so the Gaussian regression of $X(s)$ on $(X(t), \nabla X(t))$ reads

$$X(s) = r^k X(t) + r^{k-1} \sin\theta\, \langle v, \nabla X(t)\rangle + R^t(s), \tag{3.2a}$$

with a residual R^t independent of $(X(t), \nabla X(t))$. The *helix process* is

$$\zeta^t(s) := \frac{R^t(s)}{1 - r^k}, \qquad s \in M, \tag{3.2b}$$

well defined since $1 - r^k > 0$ on M. Substituting $X(t) = x > 0$ and $\nabla X(t) = 0$ in (3.2a) gives $X(s) = r^k x + (1 - r^k)\zeta^t(s)$, so $X(s) > x$ if and only if $\zeta^t(s) > x$, while the excluded points never beat t ($X(-t) = (-1)^k x \le x$). Hence, when $(X(t), \nabla X(t))$ is replaced by $(x, 0)$ in (3.2a), $x > 0$,

$$\big\{\exists s \in \mathbb{S}^{d-1} :\ X(s) > x\big\} = \Big\{\sup_M \zeta^t > x\Big\}; \tag{3.2c}$$

in this form the beating event enters the Rice formula below, since (G, ζ^t) is independent of $(X(t), \nabla X(t))$.

Proposition 3.2 (The gap as a marked Rice integral)**.** *Let k be odd and $u > 0$. Then*

$$\delta_{\mathrm{exact}}(u) - \mathbb{P}\{\sup X > u\} = C_{k,d} \int_u^\infty \Xi(x)\, \varphi(x)\, \mathrm{d}x, \tag{3.3}$$

with Ξ as in (1.3). In particular $0 \le \Xi(x) \le A(\rho x)$ and $\delta_{\mathrm{exact}}(u) - \mathbb{P}\{\sup X > u\} \le \varepsilon(u)\, \delta_{\mathrm{exact}}(u)$ with $\varepsilon(u) = \sup_{x \ge u} \Xi(x)/A(\rho x)$.

Proof. We use the Kac–Rice formula with a lower semi-continuous weight [1, Theorem 6.1], which, for a Gaussian field, requires only C^1 paths and a continuous density, bounded on compacts; it counts all zeros, degenerate ones included. We apply it on the charts of a finite atlas of $\mathbb{S}^{d-1}$ to the C^∞ Gaussian field ∇X, with $\nabla X(t) \sim N(0, kI_{d-1})$, and to the weight $g(t, X(\cdot)) = \mathbf{1}\{X(t) > u\}\mathbf{1}\{\sup_s X(s) > X(t)\}$, the indicator of an open condition on the continuous functions $(t, X(\cdot)) \mapsto X(t)$ and $\sup_s X(s) - X(t)$, hence lower semi-continuous in t and in $X(\cdot)$ for uniform convergence; the integrand $|\det \nabla^2 X|\, p_{\nabla X}(0)\, \mathrm{d}t$ is chart independent at critical points. This writes the expected number of beaten critical points above u as $\int_{\mathbb{S}^{d-1}} \mathbb{E}[|\det \nabla^2 X(t)| \mathbf{1}\{X(t) > u\}\mathbf{1}\{\sup_s X(s) > X(t)\} \mid \nabla X(t) = 0]\, p_{\nabla X(t)}(0)\, \mathrm{d}t$, where $X(t) > u$ may replace $X(t) \ge u$ in (3.1) since $\mathbb{E}N^{\mathrm{cp}}_{\{u\}} = 0$ by (1.1c). Condition further on $X(t) = x$, of density φ, independent of $\nabla X(t)$. By (1.1b), $|\det \nabla^2 X(t)| = (2k(k-1))^{(d-1)/2} |\det(\rho x I - G)|$; by (3.2c) the beating indicator is $\mathbf{1}\{\sup_M \zeta^t > x\}$; and (G, ζ^t) is independent of $(X(t), \nabla X(t))$, so the conditional expectation is an unconditional one over (G, ζ^t). Conditioning on G inside gives $\Xi(x)$, and $p_{\nabla X(t)}(0) = (2\pi k)^{-(d-1)/2}$ gives $C_{k,d}$. The bounds follow from $0 \le \mathbb{P}(\cdot \mid G) \le 1$ and (1.1c). □

Corollary 3.3 (Hölder decoupling). *For every $x > 0$ and $p > 1$, with $P(x) := \mathbb{P}\{\sup_M \zeta^t > x\}$,*

$$\Xi(x) \;\le\; \big\| \det(\rho x I - G) \big\|_p \, P(x)^{1-1/p}, \qquad \text{and} \qquad P(x) \ge \bar{\Phi}(\rho x). \tag{3.4}$$

Proof. $\Xi(x) = \mathbb{E}[|\det(\rho x I - G)|\, \mathbf{1}\{\sup_M \zeta^t > x\}]$, the mark in (1.3) being a conditional probability given G; apply Hölder's inequality. The lower bound is $\sup_M \zeta^t \ge \varsigma \lambda_{\max}(G) \ge \varsigma G_{11}$ (Lemma 3.7) and $\varsigma\rho = 1$. □

The first factor is an L^p norm of the GOE (Lemma 4.9). The second is the tail of $\sup \zeta^t$, a field of variance at most ς^2 (Lemma 3.6); by the lower bound in (3.4) it decays at best like $e^{-x^2/(2\varsigma^2)} = e^{-\rho^2 x^2/2}$, so the rate of Theorem 1.1 is the best this route can give, and with $p = \rho^2 x^2$ the power $1 - 1/p$ loses only a constant factor. The Rice formula gives the tail without the mean shift $\mathbb{E}\sup \zeta^t \approx \varsigma\sqrt{2(d-1)}$ that Borell–TIS would put into the exponent.

3.1 The layer decomposition

Let $\overline{M} := [0,\pi] \times \mathbb{S}^{d-2}$ be the compact blow-up of the sphere at $\pm t$, $M \cong (0,\pi) \times \mathbb{S}^{d-2}$; the sets $\{0\} \times \mathbb{S}^{d-2}$ and $\{\pi\} \times \mathbb{S}^{d-2}$ are the *pole fibers*.

Proposition 3.4 (Layers). *There exist independent centered Gaussian fields $Z_2, \dots, Z_k$ on $\mathbb{S}^{d-2}$, Z_j a KSS field of degree j (covariance $\langle v, v' \rangle^j$), such that*

$$\zeta^t(s) = \sum_{j=2}^{k} w_j(\theta)\, Z_j(v), \qquad w_j(\theta) = \binom{k}{j}^{1/2} \frac{r^{\,k-j}(1-r^2)^{j/2}}{1-r^k}, \qquad r = \cos\theta. \tag{3.5a}$$

Write $w(\theta) := (w_j(\theta))_{2 \le j \le k}$. The w_j are linearly independent on $(0,\pi)$ and $Z_j(v) = L_j[\zeta^t(\cdot, v)]$ for the explicit linear functionals

$$L_j[f] := \sum_{i=2}^{k} (\Gamma^{-1})_{ji} \int_0^\pi w_i(\theta) f(\theta)\, \mathrm{d}\theta, \qquad \Gamma = \Big(\int_0^\pi w_i w_j \, \mathrm{d}\theta \Big)_{2 \le i,j \le k}, \qquad L_j[w_i] = \delta_{ij}. \tag{3.5b}$$

Proof. Let $s_i = r_i t + \sqrt{1-r_i^2}\, v_i$, $c = \langle v_1, v_2 \rangle$, $q = \sqrt{(1-r_1^2)(1-r_2^2)}$. By the binomial formula $\langle s_1, s_2 \rangle^k = \sum_{j=0}^k \binom{k}{j} (r_1 r_2)^{k-j} q^j c^j$, and by (3.2a) the covariance of the regression parts is $r_1^k r_2^k + k (r_1 r_2)^{k-1} q c$, exactly the terms $j = 0, 1$. Hence $\mathbb{E}[R^t(s_1) R^t(s_2)] = \sum_{j \ge 2} \binom{k}{j} (r_1 r_2)^{k-j} q^j c^j$, which divided by $(1 - r_1^k)(1 - r_2^k)$ is the covariance of the right side of (3.5a); a centered Gaussian field is determined by its covariance. Linear independence: if $\sum_j a_j w_j \equiv 0$, multiplying by $1 - \cos^k\theta$ and dividing by $\cos^k\theta$ on $(0, \pi/2)$ gives $\sum_j a_j \binom{k}{j}^{1/2} \tan^j\theta \equiv 0$, a polynomial in $\tan\theta$ vanishing on $(0,\infty)$, so all $a_j = 0$. The w_j are bounded on $(0,\pi)$ (Lemma 3.7(i), which concerns the deterministic w_j only), so the Gram matrix Γ in $L^2(0,\pi)$ is positive definite and (3.5b) satisfies $L_j[w_i] = \delta_{ij}$. Define $Z_j(v) := L_j[\zeta^t(\cdot, v)]$: the integrals exist almost surely and commute with covariances, since $\mathbb{E} \int_0^\pi |w_i(\theta) \zeta^t(\theta, v)|\, \mathrm{d}\theta \le \int_0^\pi |w_i| \sqrt{V}\, \mathrm{d}\theta < \infty$, where $V = \mathrm{Var}\zeta^t = \sum_l w_l^2$ is bounded by the covariance identity above. From $\mathbb{E}[\zeta^t(\theta, v) \zeta^t(\theta', v')] = \sum_l w_l(\theta) w_l(\theta') \langle v, v' \rangle^l$ and $L_j[w_l] = \delta_{jl}$ one gets $\mathrm{Cov}(Z_i(v), Z_j(v')) = \delta_{ij} \langle v, v' \rangle^i$ and $\mathrm{Cov}(\zeta^t(s), Z_j(v)) = w_j(\theta)$, hence $\mathrm{Var}\big(\zeta^t(s) - \sum_j w_j(\theta) Z_j(v)\big) = V - 2\sum_j w_j^2 + \sum_j w_j^2 = 0$. So the Z_j are independent centered Gaussian fields with covariance $\langle v, v' \rangle^j$, in the Gaussian space of X; taking their continuous versions, (3.5a) holds almost surely for each s, hence for all $s \in M$ by continuity of both sides. □

Lemma 3.5 (The layer $j = 2$ is the Hessian). *Almost surely $Z_2(v) = v^\top G v$ for all $v \in \mathbb{S}^{d-2}$, with G the matrix (1.1a) at the same t; the layers $Z_3, \dots, Z_k$ are independent of G.*

Proof. From $\langle s, t \rangle^k$, $\mathrm{Cov}(X(s), \nabla^2_{ab} X(t)) = k(k-1) r^{k-2} \sin^2\theta\, v_a v_b - k r^k \delta_{ab}$, $\mathrm{Cov}(X(t), \nabla^2_{ab} X(t)) = -k\delta_{ab}$ and $\mathrm{Cov}(\nabla X(t), \nabla^2 X(t)) = 0$, so in (3.2a) the δ_{ab} terms cancel: $\mathrm{Cov}(R^t(s), \nabla^2_{ab} X(t)) = k(k -$

$1)r^{k-2}\sin^2\theta\, v_a v_b$. Dividing by $(1-r^k)\sqrt{2k(k-1)}$ and noting that the $kX(t)I$ part of G is uncorrelated with R^t, $\mathrm{Cov}(\zeta^t(s), G_{ab}) = w_2(\theta)v_a v_b$. Applying L_j of (3.5b) in θ gives $\mathrm{Cov}(Z_j(v), G_{ab}) = \delta_{j2} v_a v_b$; hence $\mathrm{Cov}(Z_2(v), v^\top G v) = \sum_{a,b} v_a^2 v_b^2 = 1$, and both variables have unit variance ($\mathrm{Var}(v^\top G v) = \sum_a v_a^4 + 2\sum_{a<b} v_a^2 v_b^2 = 1$), so they are almost surely equal, for each v and then, by continuity, for all v; the layers $j \geq 3$ are uncorrelated with G and, lying with G in the Gaussian space of X, independent of it. □

Lemma 3.6 (Variance). $V(\theta) := \mathrm{Var}\zeta^t(s) = \|w(\theta)\|^2 = \dfrac{1 - r^{2k} - k\, r^{2k-2}(1-r^2)}{(1-r^k)^2} \leq \varsigma^2 = \dfrac{2(k-1)}{k}$ *on* $(0,\pi)$, *with equality only in the limit* $\theta \to 0$ *(and* $\theta \to \pi$ *for* k *even).*

Proof. The closed form is the covariance identity of Proposition 3.4 at $s_1 = s_2$. The numerator is even in r and $(1-r^k)^2 \geq (1-|r|^k)^2$, so it suffices to take $r \in [0,1)$, where the claim reads $\psi(r) := 2(k-1)(1-r^k)^2 - k[1 - r^{2k} - k r^{2k-2}(1-r^2)] \geq 0$. Expanding, $\psi = (k-2)(1-r^k)^2 + k r^k \chi(r)$ with $\chi(r) = k r^{k-2} - (k-2) r^k - 2$, $\chi(1) = 0$, $\chi'(\varrho) = k(k-2)\varrho^{k-3}(1-\varrho^2) \leq 2k(k-2)(1-\varrho)$ on $[0,1]$, so $-\chi(r) = \int_r^1 \chi' \leq k(k-2)(1-r)^2$. The arithmetic–geometric mean inequality gives $1 - r^k = (1-r)\sum_{i<k} r^i \geq k r^{(k-1)/2}(1-r)$, hence $k r^k(-\chi) \leq k^2(k-2) r^k (1-r)^2 \leq (k-2)(1-r^k)^2$ and $\psi \geq 0$, strictly for $r \in (0,1)$; the expansion of Lemma 3.7 gives $V \to \varsigma^2$ at the pole. □

By (3.2a), $V(\theta)$ equals $\mathrm{Var}(X(s) \mid X(t), \nabla X(t))/(1-r^k)^2$, so $\varsigma^2 = \sup_\theta V$ is the constant σ_t^2 of Azaïs and Wschebor [5, Theorem 8.12] for the KSS field with derivatives along the sphere, where their κ_t vanishes because the cone of $\mathbb{S}^{d-1}$ at t is the whole tangent space, the sphere having no boundary; their exponent $1 + 1/\sigma_t^2$ is $1 + \rho^2$.

Lemma 3.7 (Poles and parities). *(i) Each* w_j *is* C^∞ *on* $[0,\pi)$, *and as* $\theta \to 0$

$$w_j(\theta) = \tfrac{2}{k}\binom{k}{j}^{1/2}\theta^{j-2}\big(1 + O(\theta^2)\big), \qquad w_2(\theta) = \varsigma\big[1 - \tfrac{k-2}{4}\theta^2 + O(\theta^4)\big],$$
$$w_3(\theta) = b_3\theta + O(\theta^3), \qquad b_3 := \tfrac{2}{k}\binom{k}{3}^{1/2} > 0.$$

For k *odd,* $w_j(\pi - \eta) = (-1)^{k-j}\tfrac{1}{2}\binom{k}{j}^{1/2}\eta^j(1 + O(\eta^2))$ *as* $\eta \to 0$; *for* k *even,* $w_j(\pi - \theta) = (-1)^{k-j} w_j(\theta)$. *(ii)* ζ^t *extends to an almost surely continuous field* $\bar\zeta(\theta, v) = \sum_j w_j(\theta) Z_j(v)$ *on* $\overline{M}$, *with* $\bar\zeta(0, v) = \varsigma v^\top G v$, *and* $\bar\zeta(\pi, \cdot) \equiv 0$ *for* k *odd,* $\bar\zeta(\pi, v) = \varsigma v^\top G v$ *for* k *even. In particular* $\sup_M \zeta^t \geq \varsigma\lambda_{\max}(G)$. *(iii) Almost surely* $Z_j(-v) = (-1)^j Z_j(v)$ *for all* v; *for* k *even,* $\zeta^t(-s) = \zeta^t(s)$.

Proof. (i) Write $1 - \cos^k\theta = \theta^2 h(\theta)$ and $\sin\theta = \theta g(\theta)$ with h, g smooth and even, $h(0) = k/2$, $g(0) = 1$; then $w_j = \binom{k}{j}^{1/2}\cos^{k-j}\theta\, g^j \theta^{j-2}/h$. The second-order term of w_2 comes from $\cos^{k-2}\theta = 1 - \frac{k-2}{2}\theta^2 + O(\theta^4)$, $\sin^2\theta = \theta^2(1 - \frac{1}{3}\theta^2 + O(\theta^4))$ and $1 - \cos^k\theta = \frac{k}{2}\theta^2(1 - \frac{3k-2}{12}\theta^2 + O(\theta^4))$. At $\theta = \pi - \eta$, $r = -\cos\eta$, $1 - r^k = 1 + \cos^k\eta \to 2$ for k odd, and for k even $r \mapsto -r$ leaves $1 - r^k$ and $1 - r^2$ invariant. (ii) The Z_j are continuous and the w_j extend continuously to $[0,\pi]$ by (i); the boundary values follow from (i) and Lemma 3.5. (iii) $\mathrm{Var}(Z_j(-v) - (-1)^j Z_j(v)) = 2 - 2(-1)^j\langle -v, v\rangle^j = 0$, and continuity; for k even, $-s = (\pi - \theta, -v)$ and $w_j(\pi - \theta)Z_j(-v) = w_j(\theta)Z_j(v)$. □

4 Interiority, the Rice bound, and the proof of Theorem 1.1

4.1 The supremum is attained in the interior

Proposition 4.1. *Let* $k \geq 3$. *Almost surely the supremum of* $\bar\zeta$ *over* $\overline{M}$ *is not attained on a pole fiber. Consequently* $\sup_M \zeta^t$ *is almost surely attained at a local maximum of* ζ^t *on* M, *and for every* $x \in \mathbb{R}$

$$\mathbf{1}\Big\{\sup_M \zeta^t > x\Big\} \leq N^{\mathrm{lm}}_{(x,\infty)}(\zeta^t) \quad a.s., \qquad \textit{hence} \qquad \mathbb{P}\Big\{\sup_M \zeta^t > x\Big\} \leq \mathbb{E}\, N^{\mathrm{lm}}_{(x,\infty)}(\zeta^t). \tag{4.1}$$

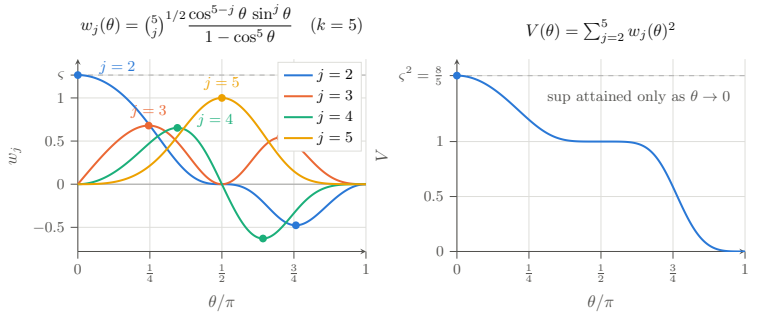


Figure 2: Layer weights w_j of (3.5a) for $k = 5$ against θ/π (left) and $V = \sum_j w_j^2$ (right). Only w_2 is maximal at the pole, where it tends to $\varsigma = \sqrt{8/5}$ (dashed); the higher layers vanish there and peak in the interior; V decreases from ς^2 to 0.

Proof. On the fiber at t, $\bar{\zeta}(0, v) = \varsigma v^\top G v$ has maximum $\varsigma \lambda_{\max}(G)$, attained exactly at $v = \pm v_1$, v_1 a unit top eigenvector: the top eigenvalue is almost surely simple, the GOE having a density and the matrices with a repeated eigenvalue forming a proper algebraic subvariety. By Lemma 3.7(i) there are constants a_j with $|w_2 - \varsigma| \le a_2\theta^2$, $|w_3 - b_3\theta| \le a_3\theta^2$ and $|w_j| \le a_j\theta^2$ for $j \ge 4$ on $[0, \pi]$, so, on an event of probability one, for all v,

$$\bar{\zeta}(\theta, v) - \varsigma v^\top G v = b_3\, \theta\, Z_3(v) + \theta^2 R(\theta, v), \qquad |R(\theta, v)| \le C(v) := 1 + \sum_j a_j |Z_j(v)| < \infty. \tag{4.2}$$

The layer Z_3 is odd (Lemma 3.7(iii)) and independent of G, hence of v_1; given G, $Z_3(v_1) \sim N(0, 1)$, so $Z_3(v_1) \neq 0$ almost surely. Let $v^* \in \{\pm v_1\}$ with $Z_3(v^*) > 0$: by (4.2), $\bar{\zeta}(\theta, v^*) > \varsigma \lambda_{\max}(G) = \max_v \bar{\zeta}(0, v)$ for $0 < \theta < b_3 Z_3(v^*)/C(v^*)$, so the supremum over $\overline{M}$ is not attained on the fiber at t. For k even the fiber at $-t$ carries the same field and $\bar{\zeta}(\pi - \theta, -v) = \bar{\zeta}(\theta, v)$, and the same argument applies. For k odd, $\bar{\zeta}(\pi, \cdot) \equiv 0$ while $\sup \bar{\zeta} > 0$ almost surely: $\lambda_{\max}(G) \neq 0$ almost surely since G has a density; if $\lambda_{\max}(G) > 0$ this is the fiber at t; if $\lambda_{\max}(G) < 0$, Lemma 3.7(i) gives $\bar{\zeta}(\pi - \eta, v) = -\frac{1}{2}\binom{k}{2}^{1/2} \eta^2 v^\top G v + O(\eta^3) > 0$ for small η. The supremum over the compact $\overline{M}$ is attained, hence at a point of M, which is a local maximum of ζ^t; on $\{\sup_M \zeta^t > x\}$ its value exceeds x, whence (4.1). □

4.2 Kac–Rice for the helix process

From now on $n = d - 2$, $s = (\theta, v) \in M$, $r = \cos\theta$, and $(e_r, e_1, \dots, e_n)$ is the orthonormal frame at s made of the meridian direction $e_r = \partial_\theta$ and of n directions tangent to the orbit sphere $\{\theta\} \times \mathbb{S}^n$, of radius $\sin\theta$. Primes are derivatives in θ, $w = (w_j)_{2 \le j \le k}$, and

$$V = \|w\|^2, \qquad C = \langle w, w' \rangle = \tfrac{1}{2} V', \qquad g_r = \|w'\|^2, \qquad \Delta = V g_r - C^2, \qquad g_t = \frac{1}{\sin^2\theta} \sum_j j\, w_j^2. \tag{4.3}$$

Lemma 4.2 (One-point law of a layer)**.** *Let Z be a KSS field of degree $j \ge 2$ on $\mathbb{S}^m$ and $v \in \mathbb{S}^m$. Then $Z(v) \sim N(0, 1)$, $\nabla Z(v) \sim N(0, j I_m)$ and $\widehat{G} := (\nabla^2 Z(v) + j Z(v) I_m)/\sqrt{2j(j-1)} \sim \mathrm{GOE}(m)$ are independent.*

Proof. The covariances of Section 1.1 with (k, d) replaced by $(j, m+1)$: $\widehat{G}$ is uncorrelated with $Z(v)$ and $\nabla Z(v)$ and has the GOE covariances. □

Lemma 4.3 (Scalars of the helix process)**.** *For $\theta \in (0, \pi)$:*
(i) $(\zeta^t(s), \partial_\theta \zeta^t(s))$ has covariance $\Sigma = \left(\begin{smallmatrix} V & C \\ C & g_r \end{smallmatrix}\right)$, and the tangential derivatives $\partial_a \zeta^t(s)$ are independent of it and of each other, each $N(0, g_t)$.

(ii) $g_t = k(1-r^{2k-2})/(1-r^k)^2$ *and* $\sum_j j(j-1)w_j^2 = k(k-1)\sin^4\theta/(1-r^k)^2$.
(iii) $\Delta = (\sin\theta\cos\theta)^{-2}\sum_{j<l}(l-j)^2 w_j^2 w_l^2$ *(by continuity at* $\pi/2$*) and*

$$\Delta \geq \frac{2^{6-2k}}{k^4} \quad \text{if } \cos\theta \geq 0 \text{ or } k \text{ even}, \qquad \Delta \geq 4^{1-k}\sin^8\theta \quad \text{if } \cos\theta \leq 0 \text{ and } k \text{ odd};$$

in particular $\Delta > 0$, $g_r \geq \Delta/V > 0$ *and* $g_t > 0$.

Proof. (i) $\zeta^t(s) = \sum_j w_j Z_j(v)$, $\partial_\theta \zeta^t(s) = \sum_j w_j' Z_j(v)$, $\partial_a \zeta^t(s) = \sin^{-1}\theta \sum_j w_j \partial_a Z_j(v)$, and Lemma 4.2. (ii) The identities $\sum_j j\binom{k}{j}a^{k-j}b^j = kb(a+b)^{k-1}$ and $\sum_j j(j-1)\binom{k}{j}a^{k-j}b^j = k(k-1)b^2(a+b)^{k-2}$ at $a = r^2$, $b = \sin^2\theta$; the terms $j = 0, 1$ of the first contribute $kr^{2k-2}\sin^2\theta$, those of the second vanish. (iii) Lagrange's identity $\Delta = \sum_{j<l}(w_j w_l' - w_l w_j')^2$ and the logarithmic derivative of (3.5a),

$$\frac{w_j'}{w_j} = j\cot\theta - (k-j)\tan\theta - g, \qquad g := \frac{k\, r^{k-1}\sin\theta}{1-r^k}, \qquad \theta \neq \tfrac{\pi}{2}, \tag{4.4}$$

give $w_j w_l' - w_l w_j' = (l-j)w_j w_l/(\sin\theta\cos\theta)$; the identities derived from (4.4) are polynomial in $(\cos\theta, \sin\theta)$ and extend to $\pi/2$ by continuity. Keeping the pairs $(2,3)$ and $(k-1,k)$,

$$\Delta \geq \max\Big(\tbinom{k}{2}\tbinom{k}{3}\frac{r^{4k-12}\sin^8\theta}{(1-r^k)^4},\ k\,\frac{\sin^{4k-4}\theta}{(1-r^k)^4}\Big).$$

If $r \geq 0$, $1-r^k = (1-r)\sum_{i<k} r^i \leq k\sin^2\theta$, so the two terms are at least r^{4k-12}/k^4 and $\sin^{4k-12}\theta/k^3$, and $\max(r^2, \sin^2\theta) \geq \frac{1}{2}$ gives $\Delta \geq k^{-4}2^{6-2k}$. If $r \leq 0$ and k is odd, $1-r^k \leq 2$, the terms are at least $r^{4k-12}\sin^8\theta/16$ and $\sin^{4k-4}\theta/16$, and $\Delta \geq \frac{\sin^8\theta}{16}\max(r^2,\sin^2\theta)^{2k-6} \geq 4^{1-k}\sin^8\theta$. For k even, Δ is symmetric about $\pi/2$ by Lemma 3.7(i). □

Lemma 4.4 (Kac–Rice for the local maxima)**.** *Let* $B = -\nabla^2\zeta^t(s)$ *and write "given* $(y,0)$*" for the conditioning on* $\zeta^t(s) = y$, $\nabla\zeta^t(s) = 0$. *For* $x > 0$,

$$\begin{aligned} \mathbb{E}\, N^{\mathrm{lm}}_{(x,\infty)}(\zeta^t) &= \omega_n \int_0^\pi \sin^n\theta \int_x^\infty p_\theta(y)\, \mathbb{E}\big[\det B\, \mathbf{1}\{B \succ 0\} \,\big|\, (y,0)\big]\, \mathrm{d}y\, \mathrm{d}\theta, \\ p_\theta(y) &= \frac{\Delta^{-1/2}\, g_t^{-n/2}}{(2\pi)^{d/2}}\, e^{-y^2 g_r/(2\Delta)}, \end{aligned} \tag{4.5}$$

with $\omega_n = 2\pi^{(n+1)/2}/\Gamma(\frac{n+1}{2})$ *the volume of* $\mathbb{S}^n$ *and* $p_\theta(y)$ *the density of* $(\zeta^t(s), \nabla\zeta^t(s))$ *at* $(y,0)$.

Proof. The paths of ζ^t are C^∞ on M. By Lemma 4.3, $\nabla\zeta^t(s)$ has the nondegenerate covariance $\mathrm{diag}(g_r, g_t I_n)$, continuous in s, so its density is continuous and bounded on compact subsets of M, and the Kac–Rice formula with lower semi-continuous weights of the proof of Proposition 3.2 applies to $\nabla\zeta^t$ on the charts of M, with weights that are indicators of open conditions on $(s, \zeta^t(\cdot), \nabla^2\zeta^t(\cdot))$. First, for a compact $K \subset M$ and $\epsilon > 0$, the weight $\mathbf{1}\{|\det\nabla^2\zeta^t(s)| < \epsilon\}$ gives $\mathbb{E}\#\{s \in K : \nabla\zeta^t(s) = 0,\ |\det\nabla^2\zeta^t(s)| < \epsilon\} \leq \epsilon\int_K p_{\nabla\zeta^t(s)}(0)\,\mathrm{d}s \to 0$ as $\epsilon \to 0$; exhausting M by compacts, almost surely no critical point of ζ^t in M is degenerate, so every local maximum has $B \succ 0$. Second, the weight $\mathbf{1}\{\nabla^2\zeta^t(s) \prec 0\}\mathbf{1}\{\zeta^t(s) > x\}$, on compacts exhausting M, gives $\mathbb{E}N^{\mathrm{lm}}_{(x,\infty)}(\zeta^t) = \int_M \mathbb{E}[|\det\nabla^2\zeta^t|\mathbf{1}\{\nabla^2\zeta^t \prec 0\}\mathbf{1}\{\zeta^t > x\} \mid \nabla\zeta^t = 0]\, p_{\nabla\zeta^t(s)}(0)\,\mathrm{d}s$. The law of the field around s depends on s only through θ (the layers $j \geq 3$ are isotropic and $Z_2(Rv) = v^\top(R^\top G R)v$ with $R^\top G R \sim \mathrm{GOE}(d-1)$), so $\int_M \mathrm{d}s = \omega_n\int_0^\pi \sin^n\theta\,\mathrm{d}\theta$; conditioning on $\zeta^t(s) = y$ and Lemma 4.3(i) give the density, with $\det\Sigma = \Delta$ and $(\Sigma^{-1})_{11} = g_r/\Delta$. □

4.3 The diagonal of the conditional Hessian

Lemma 4.5. *Given $(y,0)$, the entries $B_{11},\dots,B_{nn},B_{rr}$ of $B=-\nabla^2\zeta^t(s)$ are independent Gaussian variables,*

$$B_{aa}\sim N\big(\mu_t\,y,\ 2\beta_{tt}\big),\qquad \mu_t=\frac{k}{1-r^k},\qquad \beta_{tt}=\frac{k(k-1)}{(1-r^k)^2},\qquad \frac{\mu_t}{\sqrt{2\beta_{tt}}}=\rho,$$

and $\mathbb{E}[B_{rr}^2\mid(y,0)]\le\|w''\|^2\big(1+y^2g_r/\Delta\big)$.

Proof. In the coordinates (θ,v) the round metric is the warped product $\mathrm{d}\theta^2+\sin^2\theta\,h$, and for a unit field $\widetilde U$ of $(\mathbb{S}^n,h)$, $\nabla_{\widetilde U}\widetilde U=\nabla^h_{\widetilde U}\widetilde U-\sin\theta\cos\theta\,\partial_\theta$, $\nabla_{\partial_\theta}\partial_\theta=0$, $e_a=\widetilde U/\sin\theta$. Hence, for $f=\sum_j w_jZ_j$,

$$(\nabla^2 f)_{rr}=\sum_j w_j''Z_j(v),\qquad (\nabla^2 f)_{aa}=\frac{1}{\sin^2\theta}\sum_j w_j\,(\nabla^2_{\mathbb{S}^n}Z_j)_{aa}(v)+\cot\theta\sum_j w_j'Z_j(v).$$

For $f=\cos\theta$ and $f=\sin\theta\,v_e$, restrictions of linear forms, whose Hessian on the sphere is $-f$ times the metric, both formulas give $-f$. By Lemma 4.2, $(\nabla^2_{\mathbb{S}^n}Z_j)_{aa}=\sqrt{2j(j-1)}\widehat G^{(j)}_{aa}-jZ_j(v)$ with independent $\widehat G^{(j)}\sim\mathrm{GOE}(n)$, independent of all $Z_j(v)$, $\nabla Z_j(v)$. So

$$B_{aa}=\sum_j\Big[\frac{j\,w_j}{\sin^2\theta}-\cot\theta\,w_j'\Big]Z_j(v)-\sum_j\frac{\sqrt{2j(j-1)}\,w_j}{\sin^2\theta}\,\widehat G^{(j)}_{aa}.$$

Multiplying (4.4) by $\sin\theta\cos\theta\,w_j$ gives $jw_j=\sin\theta\cos\theta\,w_j'+(k\sin^2\theta+g\sin\theta\cos\theta)w_j$, so the bracket equals $(k+g\cot\theta)w_j=\frac{k}{1-r^k}w_j$ ($g\cot\theta=kr^k/(1-r^k)$): the first sum is $\mu_t\zeta^t(s)=\mu_t y$. The second is centered Gaussian, independent of the local data, of variance $\sin^{-4}\theta\sum_j 2j(j-1)w_j^2=2\beta_{tt}$ (Lemma 4.3(ii)), and independent across a. Given $(y,0)$, each B_{aa} is therefore $\mu_t y$ plus a centered Gaussian variable built on $(\widehat G^{(j)}_{aa})_{2\le j\le k}$, while B_{rr} below is built on the $Z_j(v)$ alone; the two families are independent, which gives the joint independence of $B_{11},\dots,B_{nn},B_{rr}$ given $(y,0)$. Next $B_{rr}=-\sum_j w_j''Z_j(v)$ is a function of the $Z_j(v)$, hence independent of the $\widehat G^{(j)}$, and unaffected by the conditioning on $\partial_a\zeta^t(s)=0$. Given $(\zeta^t(s),\partial_\theta\zeta^t(s))=(y,0)$ it is Gaussian with variance at most $\mathrm{Var}B_{rr}=\|w''\|^2$ and mean $\gamma^\top\Sigma^{-1}(y,0)^\top$, $\gamma=\mathrm{Cov}(B_{rr},(\zeta^t(s),\partial_\theta\zeta^t(s)))$, bounded by $\sqrt{\gamma^\top\Sigma^{-1}\gamma}\sqrt{(\Sigma^{-1})_{11}}\,y\le\|w''\|\sqrt{g_r/\Delta}\,y$, since $\gamma^\top\Sigma^{-1}\gamma\le\mathrm{Var}B_{rr}$ is the explained variance. □

Lemma 4.6 (Bounds on the weights)**.** *For $\theta\in(0,\pi)$: (i) $\|w''(\theta)\|\le W_k$, where $W_k:=9k^3$ for k even and $W_k:=43k^3$ for k odd; (ii) if k is odd and $\cos\theta\le0$, $V(\theta)\le2^k\sin^4\theta$.*

Proof. With $S_k(r)=\sum_{i<k}r^i=(1-r^k)/(1-r)$ and $\sin^2\theta=(1-r)(1+r)$, for $j\ge2$

$$w_j(\theta)=s_k(\theta)\,T_j(\theta),\qquad s_k(\theta):=\frac{1}{S_k(\cos\theta)},\qquad T_j(\theta):=\binom{k}{j}^{1/2}\cos^{k-j}\theta\,(1+\cos\theta)\,\sin^{j-2}\theta,\tag{4.6}$$

a form with no singularity at $\theta=0$. Write $T=(T_j)_{2\le j\le k}$, so that $w=s_kT$ and

$$\|w''\|\le s_k\|T''\|+2|s_k'|\,\|T'\|+|s_k''|\,\|T\|.$$

The vector T. With $p=\sin^2\theta$, $\sum_{j\ge2}\binom{k}{j}\cos^{2(k-j)}\theta\sin^{2j}\theta=\mathbb{P}\{\mathrm{Bin}(k,p)\ge2\}\le\binom{k}{2}p^2$, so $\|T(\theta)\|^2=(1+\cos\theta)^2\sin^{-4}\theta\sum_{j\ge2}\binom{k}{j}\cos^{2(k-j)}\theta\sin^{2j}\theta\le4\binom{k}{2}$ for every $\theta\in\mathbb{R}$, hence $\sup_{\mathbb{R}}\|T\|\le\sqrt{2k(k-1)}$. Each T_j is a trigonometric polynomial of degree $k-1$, and Bernstein's inequality [8, Chapter 4] holds for $\mathbb{R}^{k-1}$-valued trigonometric polynomials with the same constant (for fixed θ_0, apply it to $\langle T(\cdot),u\rangle$ with $u=T'(\theta_0)/\|T'(\theta_0)\|$); hence $\|T'\|\le(k-1)\sqrt{2k(k-1)}$ and $\|T''\|\le(k-1)^2\sqrt{2k(k-1)}$.

The scalar s_k for $\cos\theta \ge 0$. On $[0,1]$, $S_k \ge 1$, $S_k' = \sum_{i<k} i\, r^{i-1} \le (k-1)S_k$ and $S_k'' \le (k-1)(k-2)S_k$. From $s_k' = S_k' \sin\theta / S_k^2$ and $s_k'' = (S_k' \cos\theta - S_k'' \sin^2\theta)/S_k^2 + 2S_k'^2 \sin^2\theta / S_k^3$ we get $s_k \le 1$, $|s_k'| \le k-1$, $|s_k''| \le (k-1) + (k-1)(k-2) + 2(k-1)^2 = 3(k-1)^2$, and

$$\|w''\| \le \sqrt{2k(k-1)}\,\big[(k-1)^2 + 2(k-1)^2 + 3(k-1)^2\big] = 6(k-1)^2\sqrt{2k(k-1)} \le 9k^3.$$

For k even, $\|w''(\pi-\theta)\| = \|w''(\theta)\|$ by Lemma 3.7(i), which proves (i) in that case.

The scalar s_k for $\cos\theta < 0$, k odd. Put $r = -x$, $x \in (0,1]$. Then $S_k(r) = (1+x^k)/(1+x) \in [\tfrac12, 1]$, $S_k'(r) = n(x)/(1+x)^2$ with $n(x) = 1 - kx^{k-1} - (k-1)x^k$ decreasing from 1 to $2-2k$, and $S_k''(r) = k(k-1)x^{k-2}/(1+x) + 2S_k'(r)/(1+x)$. If $n(x) < 0$ then $(2k-1)x^{k-1} > 1$, so $x \ge 5^{-1/2}$ and $(1+x)^2 > 2$; hence $|S_k'| \le k-1$ and $|S_k''| \le k(k-1) + 2(k-1) \le k(k+1)$. Therefore $s_k \le 2$, $|s_k'| \le 4(k-1)$, $|s_k''| \le 4\big(k(k+1) + k - 1\big) + 16(k-1)^2 \le 20k^2$, and

$$\|w''\| \le \sqrt{2k(k-1)}\,\big[2(k-1)^2 + 8(k-1)^2 + 20k^2\big] \le 30k^2\sqrt{2k(k-1)} \le 43k^3.$$

(ii) For k odd and $r \le 0$, $1 - r^k \ge 1$, so $w_j^2 \le \binom{k}{j}\sin^{2j}\theta \le \binom{k}{j}\sin^4\theta$. □

4.4 The Rice bound

Lemma 4.7 (Gaussian tail integrals)**.** *For $m \ge 0$ and $a > 0$ with $a(1+a) \ge 2m$, $\int_a^\infty (1+z)^m e^{-z^2/2}\,\mathrm{d}z \le 2\,(1+a)^m a^{-1} e^{-a^2/2}$, and $a \mapsto (1+a)^m e^{-a^2/2}$ is decreasing on $\{a(1+a) \ge m\}$. For $m = 0$ the factor 2 can be dropped: $\bar\Phi(a) \le \varphi(a)/a$.*

Proof. Let J be the integral. Since $1 \le z/a$ on $[a,\infty)$, integrating by parts,

$$\begin{aligned} J \le \frac1a \int_a^\infty z(1+z)^m e^{-z^2/2}\,\mathrm{d}z &= \frac{(1+a)^m e^{-a^2/2}}{a} + \frac{m}{a}\int_a^\infty (1+z)^{m-1} e^{-z^2/2}\,\mathrm{d}z \\ &\le \frac{(1+a)^m e^{-a^2/2}}{a} + \frac{m}{a(1+a)} J, \end{aligned}$$

whence the claim; the logarithmic derivative of $(1+a)^m e^{-a^2/2}$ is $\frac{m}{1+a} - a$. □

Proposition 4.8 (Rice bound with explicit constant)**.** *For $\rho x \ge \sqrt{2(d-1)}$,*

$$\mathbb{P}\Big\{\sup_M \zeta^t > x\Big\} \;\le\; \mathbb{E}\, N^{\mathrm{lm}}_{(x,\infty)}(\zeta^t) \;\le\; \bar P(x) := \frac{\sqrt{\pi}\, W_k\, 2^{2k}\,(k-1)^{(d-2)/2}}{\Gamma\big(\frac{d-1}{2}\big)}\,\frac{(1+\rho x)^{d-1}}{\rho x}\, e^{-\rho^2 x^2/2}, \tag{4.7}$$

with W_k as in Lemma 4.6; the constant is $C^{\mathrm{Rice}}_{k,d}$ of (1.4) for k odd and $2C^{\mathrm{Rice}}_{k,d}$ for k even.

Proof. The first inequality is (4.1). On $\{B \succ 0\}$ Hadamard's inequality gives $0 < \det B \le \prod_i B_{ii}$, so $\det B\,\mathbf{1}\{B \succ 0\} \le \prod_i (B_{ii})_+$, and by Lemma 4.5

$$\mathbb{E}\big[\det B\,\mathbf{1}\{B \succ 0\} \mid (y,0)\big] \le \mathbb{E}[(B_{rr})_+ \mid (y,0)] \prod_{a=1}^{n} \mathbb{E}[(B_{aa})_+ \mid (y,0)] =: H_\theta(y),$$

with $\mathbb{E}[(B_{aa})_+ \mid (y,0)] = \sqrt{2\beta_{tt}}\,\varpi(\rho y)$, $\varpi(u) = u\Phi(u) + \varphi(u) \le u+1$ for $u \ge 0$, and $\mathbb{E}[(B_{rr})_+ \mid (y,0)] \le \|w''\|(1 + y^2 g_r/\Delta)^{1/2}$. Put $\varkappa := \sqrt{\Delta/g_r}$; then $\varkappa^2 = V - C^2/g_r \le V \le \varsigma^2$ (Lemma 3.6), so $\rho\varkappa \le 1$. Substituting $y = \varkappa z$ in the inner integral of (4.5): $\Delta^{-1/2}\varkappa = g_r^{-1/2}$, $y^2 g_r/\Delta = z^2$, $\rho y \le z$, $(1+z^2)^{1/2} \le 1+z$, and the lower limit is $x/\varkappa \ge \rho x$; hence

$$\int_x^\infty p_\theta(y) H_\theta(y)\,\mathrm{d}y \le \frac{(2\beta_{tt})^{n/2}}{(2\pi)^{d/2} g_t^{n/2}} \cdot \frac{\|w''\|}{\sqrt{g_r}}\, J\Big(\frac{x}{\varkappa}\Big), \qquad J(a) := \int_a^\infty (1+z)^{d-1} e^{-z^2/2}\,\mathrm{d}z.$$

By Lemma 4.3(ii), $\sin^2\theta\,\beta_{tt}/g_t = (k-1)/\sum_{i=0}^{k-2} r^{2i} \le k-1$, so $\sin^n\theta(2\beta_{tt}/g_t)^{n/2} \le (2(k-1))^{n/2}$ and

$$\mathbb{E}\,N^{\mathrm{lm}}_{(x,\infty)}(\zeta^t) \le \frac{\omega_n(2(k-1))^{n/2}}{(2\pi)^{d/2}}\,W_k\int_0^\pi \frac{J(x/\varkappa(\theta))}{\sqrt{g_r(\theta)}}\,\mathrm{d}\theta. \tag{4.8}$$

Away from the antipode (all θ if k is even, $\cos\theta \ge 0$ if k is odd), Lemmas 4.3(iii) and 3.6 give $g_r \ge \Delta/V \ge 2^{5-2k}/k^4$, and Lemma 4.7 with $m = d-1$ (its hypothesis $\rho x(1+\rho x) \ge 2(d-1)$ holds) gives $J(x/\varkappa) \le J(\rho x) \le 2(1+\rho x)^{d-1}(\rho x)^{-1}e^{-\rho^2x^2/2}$: the integrand of (4.8) is at most $k^2 2^{k-3/2}\,(1+\rho x)^{d-1}(\rho x)^{-1}e^{-\rho^2x^2/2}$. Near the antipode ($k$ odd, $\cos\theta < 0$), Lemmas 4.3(iii) and 4.6(ii) give $1/\sqrt{g_r} \le \sqrt{V/\Delta} \le 2^{3k/2-1}/\sin^2\theta$ and $\varkappa \le \sqrt{V} \le 2^{k/2}\sin^2\theta$, hence $1/\sqrt{g_r} \le 2^{2k-1}/\varkappa$; with $a = x/\varkappa \ge \rho x$, Lemma 4.7 gives $\varkappa^{-1}J(x/\varkappa) \le \frac{2}{x}(1+a)^{d-1}e^{-a^2/2} \le \frac{2}{x}(1+\rho x)^{d-1}e^{-\rho^2x^2/2}$, so the integrand is at most $2^{2k}\rho\,(1+\rho x)^{d-1}(\rho x)^{-1}e^{-\rho^2x^2/2}$. As $k^2 2^{k-3/2} \le 2^{2k}$ and $\rho < 1$, the integral in (4.8) is at most $\pi 2^{2k}(1+\rho x)^{d-1}(\rho x)^{-1}e^{-\rho^2x^2/2}$, and $\omega_n(2(k-1))^{n/2}(2\pi)^{-d/2} = (k-1)^{(d-2)/2}/(\sqrt{\pi}\,\Gamma(\frac{d-1}{2}))$ give (4.7). □

4.5 Two moments of the GOE and the proof of Theorem 1.1

Lemma 4.9 (Two facts on the GOE). *Let $G \sim \mathrm{GOE}(N)$, $N \ge 2$, $\nu > 0$.*

(i) $\mathbb{E}\det(\nu I - G) = \sum_{i=0}^{\lfloor N/2\rfloor} \frac{(-1)^i N!}{i!\,(N-2i)!\,4^i}\,\nu^{N-2i}$, *and* $\mathbb{E}|\det(\nu I-G)| \ge \mathbb{E}\det(\nu I-G) \ge \frac{3}{4}\nu^N$ *for* $\nu \ge N$.

(ii) $\mathbb{P}\{\|G\| > \sqrt{2N}+t\} \le 2e^{-t^2/2}$ *for* $t \ge 0$; *hence* $\|\,\|G\|\,\|_q \le \sqrt{2N} + \frac{5}{6}\sqrt{q}$ *for* $q \ge 8$, *and for* $p \ge 8/N$

$$\big\|\det(\nu I - G)\big\|_p \;\le\; \Big(\nu + \sqrt{2N} + \tfrac{5}{6}\sqrt{Np}\Big)^N.$$

Proof. (i) $\det(\nu I - G) = \sum_j (-1)^j e_j(G)\nu^{N-j}$, $e_j(G)$ the sum of the principal $j\times j$ minors, each a $\mathrm{GOE}(j)$ matrix G'. In $\mathbb{E}\det G' = \sum_\pi \mathrm{sgn}(\pi)\mathbb{E}\prod_i G'_{i\pi(i)}$ the entries are centered and independent up to symmetry, so a term survives only if every entry appears an even number of times, i.e. π is a product of $j/2$ disjoint transpositions; each such π contributes $(-1)^{j/2}2^{-j/2}$ and there are $(j-1)!!$ of them. Hence $\mathbb{E}e_j(G) = \binom{N}{j}(-1)^{j/2}(j-1)!!2^{-j/2}$ for j even, 0 for j odd, and $\binom{N}{2i}(2i-1)!!2^{-i} = N!/(i!(N-2i)!4^i)$. The terms $t_i = \frac{N!}{i!(N-2i)!}(4\nu^2)^{-i}\nu^N$ satisfy $t_{i+1}/t_i \le N^2/(4\nu^2) \le \frac{1}{4}$ for $\nu \ge N$, so the alternating sum is at least $t_0 - t_1 \ge \frac{3}{4}\nu^N$. (ii) $\lambda_{\max}(G) = \sup_{|v|=1} v^\top G v$ is the supremum of a centered Gaussian process of unit variance, and $\mathrm{Var}(v^\top Gv - u^\top Gu) = 2(1-\langle u,v\rangle^2) \le 4(1-\langle u,v\rangle) = \mathrm{Var}(\sqrt{2}\langle\xi,v\rangle - \sqrt{2}\langle\xi,u\rangle)$ for $\xi \sim N(0,I_N)$, so the Sudakov–Fernique inequality gives $\mathbb{E}\lambda_{\max}(G) \le \sqrt{2}\,\mathbb{E}\|\xi\| \le \sqrt{2N}$ and Borell–TIS gives $\mathbb{P}\{\lambda_{\max}(G) > \sqrt{2N}+t\} \le e^{-t^2/2}$; the factor 2 comes from $\|G\| = \max(\lambda_{\max}(G), \lambda_{\max}(-G))$ and $-G \sim G$. Then $\mathbb{E}(\|G\| - \sqrt{2N})_+^q = \int_0^\infty q t^{q-1}\mathbb{P}\{\|G\| > \sqrt{2N}+t\}\,\mathrm{d}t \le 2q\int_0^\infty t^{q-1}e^{-t^2/2}\,\mathrm{d}t = q\,2^{q/2}\Gamma(q/2) \le 2\sqrt{\pi q}\,(q/e)^{q/2}e^{1/(6q)}$ by Stirling's bound $\Gamma(z) \le \sqrt{2\pi}\,z^{z-1/2}e^{-z}e^{1/(12z)}$; the q-th root is $(2\sqrt{\pi q})^{1/q}e^{1/(6q^2)}\sqrt{q/e} \le 0.82\sqrt{q} \le \frac{5}{6}\sqrt{q}$ for $q \ge 8$, the prefactor being decreasing in q; Minkowski gives $\|\,\|G\|\,\|_q \le \sqrt{2N} + \frac{5}{6}\sqrt{q}$. Finally $|\det(\nu I - G)| \le (\nu + \|G\|)^N$, so $\|\det(\nu I - G)\|_p \le \|\nu + \|G\|\,\|_{Np}^N \le (\nu + \|\,\|G\|\,\|_{Np})^N$ by Minkowski in L^{Np}, with $q = Np \ge 8$. □

Proof of Theorem 1.1, k odd. The upper bound is Lemma 3.1, and $\delta_{\mathrm{exact}}(u) - \mathbb{P}\{\sup X > u\} \le \varepsilon(u)\delta_{\mathrm{exact}}(u)$ is Proposition 3.2. Let $\rho u \ge \min\{d-1, \frac{3}{2}\sqrt{2(d-1)}\}$ and $x \ge u$; put $N := d-1 \ge 2$ and $a := \rho x$. Then $a \ge 2$, $\sqrt{2N} \le a$, $Na^2 \ge 8$ and $a(1+a) \ge N$. Take $p = a^2$ in Corollary 3.3. By Lemma 4.9(ii), $\|\det(\rho x I - G)\|_p \le (a + \sqrt{2N} + \frac{5}{6}\sqrt{N}\,a)^N \le a^N(2 + \frac{5}{6}\sqrt{N})^N$, since $\sqrt{2N} \le a$. By the lower bound in (3.4) and $\bar\Phi(a) \ge \frac{a}{1+a^2}\varphi(a)$,

$$P(x)^{-1/p} \le \Big(\frac{\sqrt{2\pi}\,(1+a^2)}{a}\Big)^{1/a^2} e^{1/2} \le 3 \qquad (a \ge 2),$$

the left side being decreasing in a and equal to 2.61 at $a = 2$; hence $P(x)^{1-1/p} \le 3P(x) \le 3\bar{P}(x)$ by Proposition 4.8. If $\rho u \ge d-1$, then $a \ge N$ and $A(\rho x) \ge \frac{3}{4}(\rho x)^N$ (Lemma 4.9(i)), so

$$\frac{\Xi(x)}{A(\rho x)} \;\le\; 3 \cdot \tfrac{4}{3}\Big(2 + \tfrac{5}{6}\sqrt{d-1}\Big)^{d-1} \bar{P}(x) = C_{k,d}^{\mathrm{gap}} \, \frac{(1+\rho x)^{d-1}}{\rho x} \, e^{-\rho^2 x^2/2}.$$

If $\frac{3}{2}\sqrt{2N} \le \rho u < N$, write $a = e_x\sqrt{2N}$ with $e_x \ge \frac{3}{2}$; then (A.6) of Appendix A gives $A(\rho x) \ge (a^2 - 2N - 1)^{N/2} = a^N(1-e_x^{-2})^{N/2}\big(1 - \frac{1}{2N(e_x^2-1)}\big)^{N/2} \ge \frac{4}{5}\big(\frac{5}{9}\big)^{N/2} a^N$ by Bernoulli's inequality, and the factor $3 \cdot \frac{4}{3}$ becomes $3 \cdot \frac{5}{4}(\frac{9}{5})^{N/2} = \frac{15}{4}(\frac{9}{5})^{N/2}$. In both cases the bound is decreasing in x on $[u,\infty)$ (the logarithmic derivative of $(1+a)^{d-1}a^{-1}e^{-a^2/2}$ is $\frac{d-1}{1+a} - \frac{1}{a} - a < 0$ for $a(1+a) \ge d-1$); the supremum over $x \ge u$ is at $x = u$, which gives the two bounds of the theorem. □

Proposition 4.10 (k even)**.** *Let k be even. Then $\mathbb{P}\{\sup X > u\} \le \delta_*(u) = \frac{1}{2}\delta_{\mathrm{exact}}(u)$ and $\delta_*(u) - \mathbb{P}\{\sup X > u\} = \frac{1}{2}C_{k,d}\int_u^\infty \Xi(x)\varphi(x)\,\mathrm{d}x$ for $u > 0$. Moreover $\mathbf{1}\{\sup_M \zeta^t > x\} \le \frac{1}{2}N^{\mathrm{lm}}_{(x,\infty)}(\zeta^t)$ almost surely, so $\mathbb{P}\{\sup_M \zeta^t > x\} \le \frac{1}{2}\bar{P}(x)$ for $\rho x \ge \sqrt{2(d-1)}$, and Theorem 1.1 holds as stated.*

Proof. $X(-s) = X(s)$ and the antipodal map is an isometry, so critical points come in antipodal pairs with equal values and Hessians and $N^{\mathrm{cp}}_{[u,\infty)}$ is even. On $\mathbb{RP}^{d-1}$ the field $\bar{X}([s]) = X(s)$ has $\mathrm{Var}(\bar{X}([s]) - \bar{X}([t])) = 2(1 - \langle s,t\rangle^k) > 0$ for $[s] \ne [t]$, so its maximum is almost surely attained at a single point [13, Theorem 3], and $\mathbb{P}\{\sup X = u\} \le \mathbb{E}N^{\mathrm{cp}}_{\{u\}} = 0$ by (1.1c). Hence $N^{\mathrm{cp}}_{[u,\infty)} = 2\mathbf{1}\{\sup X \ge u\} + \#\{\text{beaten critical points above } u\}$, and the marked Rice formula of Proposition 3.2, whose proof uses the parity of k only through (3.2c) ($X(-t) = x$ never beats t strictly), gives the identity. If $s_* \in M$ is the interior local maximum at which $\sup_M \zeta^t$ is attained (Proposition 4.1), then $-s_* \in M$, $\zeta^t(-s) = \zeta^t(s)$ for all $s \in M$ (Lemma 3.7(iii)), so $-s_* \ne s_*$ is a local maximum with the same value: on $\{\sup_M \zeta^t > x\}$, $N^{\mathrm{lm}}_{(x,\infty)}(\zeta^t) \ge 2$. Sections 3.1 to 4.4 hold for k even without change (Proposition 4.1 treats both fibers), so $\mathbb{P}\{\sup_M \zeta^t > x\} \le \frac{1}{2}\mathbb{E}N^{\mathrm{lm}}_{(x,\infty)}(\zeta^t) \le \frac{1}{2}\bar{P}(x)$ by Proposition 4.8, and the proof of Theorem 1.1 applies with $\bar{P}$ replaced by $\frac{1}{2}\bar{P}$, which is the factor $\frac{1}{2}$ in $C^{\mathrm{Rice}}_{k,d}$ for k even. □

Numerical illustration. Table 2 lists, in the range $\rho u \ge d-1$ of the theorem, the Rice integral (4.5) with Hadamard's bound H_θ evaluated numerically, its explicit bound $\bar{P}(u)$, and the two resulting values of the bound on $\Xi(u)/A(\rho u)$ from the proof of Theorem 1.1: with the numerical integral, and with the explicit bound (1.4) (scripts in `github.com/ydecastro/TENSOR-KSS`). The explicit $C^{\mathrm{Rice}}_{k,d}$ is crude by a factor between $3 \cdot 10^6$ and $9 \cdot 10^6$, which comes from the uniform bounds W_k and the lower bounds on Δ (W_k alone is loose by a factor of order k^2: numerically $\sup_\theta \|w''(\theta)\| \le 2k$ for $3 \le k \le 10$); the factor $3(2 + \frac{5}{6}\sqrt{d-1})^{d-1}$ of the proof is $5.4 \cdot 10^2$ for $d = 5$ and $2.6 \cdot 10^3$ for $d = 6$. With the numerical integral the bracket is non-vacuous from 1.51 to 1.56 u_{IMF} on, $u_{\mathrm{IMF}} = \sqrt{2d-1}/\rho$, with the explicit constant from 2.33 to 2.40 u_{IMF} on.

(k,d)	u	u/u_{IMF}	Rice integral (4.5)	$\bar{P}(u)$ ($\frac{1}{2}\bar{P}$ for $k=4$)	ε, numerical	ε, (1.4)
$(3,5)$	8.66	2.5	$4.8 \cdot 10^{-11}$	$1.6 \cdot 10^{-4}$	$2.8 \cdot 10^{-8}$	0.12
	13.86	4.0	$1.0 \cdot 10^{-29}$	$4.8 \cdot 10^{-23}$	$5.6 \cdot 10^{-27}$	$3.5 \cdot 10^{-20}$
$(3,6)$	9.57	2.5	$7.1 \cdot 10^{-13}$	$3.9 \cdot 10^{-6}$	$2.0 \cdot 10^{-9}$	0.014
	15.32	4.0	$1.4 \cdot 10^{-35}$	$1.1 \cdot 10^{-28}$	$3.7 \cdot 10^{-32}$	$3.7 \cdot 10^{-25}$
$(4,6)$	10.16	2.5	$1.4 \cdot 10^{-12}$	$8.8 \cdot 10^{-6}$	$2.0 \cdot 10^{-9}$	0.030
	16.25	4.0	$2.8 \cdot 10^{-35}$	$2.4 \cdot 10^{-28}$	$3.7 \cdot 10^{-32}$	$8.3 \cdot 10^{-25}$

Table 2: Numerical against explicit; "ε, numerical" is the bound of the proof of Theorem 1.1 on $\Xi(u)/A(\rho u)$ with the Rice integral evaluated numerically. For $k = 4$, $\bar{P}$ and ε include the factor $\frac{1}{2}$ of Proposition 4.10. All entries are rounded up.

5 A non-asymptotic annealed complexity

The Kostlan–Shub–Smale field X coincides with the Hamiltonian of the (centered, isotropic) spherical k-spin model from spin-glass physics [2, 7]. Its critical-point structure, the count of local maxima above an energy level E, is a central object of study. Theorem 5.1 concerns the annealed complexity, the first moment of the count; the quenched complexity is studied through the second moment by Subag [17], and the critical points near the maximum by Subag and Zeitouni [18]. Auffinger et al. [2] write $\Theta_{0,p}(u)$ for the growth rate of the number of local minima of the order-p model below the level Nu, and Θ_p for the rate of all critical points. By the symmetry $H \overset{d}{=} -H$, the rate of local maxima above Nu is $\Theta_{0,p}(-u)$, and $\Theta_{0,p} = \Theta_p$ on $u \le -E_\infty$ [2, Eqs. (2.15) and (2.16)]. We write $\Theta_k := \Theta_{0,p}\big|_{p=k}$. Here $E_\infty := 2\sqrt{(k-1)/k}$ is the spectral-edge energy [2, Eq. (2.13)], with $\rho E_\infty = \sqrt{2}$, and $E_0 > E_\infty$ is the ground-state energy above which no critical point survives [2, Eq. (2.22) and Theorem 2.12]. In our normalization the level E corresponds to $u_{\mathrm{ABC}} = E/\sqrt{d}$ (Remark 5.5). Theorem 2.4 and Lemma 4.9 turn the asymptotic statement $\lim_{d\to\infty} d^{-1}\log \mathbb{E}[N^{\mathrm{lm}}_{[E,\infty)}] = \Theta_k(-u_{\mathrm{ABC}})$ into a non-asymptotic two-sided bracket at every finite (k, d).

Theorem 5.1 (Non-asymptotic annealed complexity)**.** *Let $k, d \ge 3$. For $E > 0$ put*

$$e \;:=\; \frac{\rho E}{\sqrt{2(d-1)}}\,, \qquad \kappa(e) \;:=\; \frac{2e\,\exp(-(e-1)^2)}{\sqrt{e^2-1}} \quad (e>1)\,. \tag{5.1a}$$

(i) *For every $E \in \mathbb{R}$,*

$$\mathbb{E}\big[N^{\mathrm{lm}}_{[E,\infty)}\big] \;\le\; \mathbb{E}\big[N^{\mathrm{cp}}_{[E,\infty)}\big] \;=\; \delta_{\mathrm{exact}}(E)\,, \tag{5.1b}$$

where $N^{\mathrm{cp}}_{[E,\infty)}$ counts all critical points with $X(s) \ge E$.

(ii) *For every E with $e \ge 3/2$,*

$$\big(1 - 4\,\kappa(e)^{d-1}\big)\,\delta_{\mathrm{exact}}(E) \;\le\; \mathbb{E}\big[N^{\mathrm{lm}}_{[E,\infty)}\big] \;\le\; \delta_{\mathrm{exact}}(E)\,. \tag{5.1c}$$

The function κ is decreasing on $[3/2,\infty)$ and $\kappa(e) < 1$ if and only if $e > e_* = 1.9224\ldots$; thus the lower bound in (5.1c) is non-trivial for $e \ge 9/4$ at every $d \ge 3$, for $e \ge 2$ when $d \ge 10$ ($\kappa(2) = 4\exp(-1)/\sqrt{3} = 0.850$), and for every fixed $e > e_*$ when d is large. By Step 3b of the proof in Appendix A, $E/\sqrt{d} \to eE_\infty$ as $d \to \infty$: e is the energy in units of the spectral edge.

Corollary 5.2 (Annealed complexity of deep local minima, by sign symmetry)**.** *Let $N^{\mathrm{lm,min}}_{(-\infty,E']}$ denote the number of local minima of X with value at most E'.*

(i′) *For every $E' \in \mathbb{R}$,*

$$\mathbb{E}\big[N^{\mathrm{lm,min}}_{(-\infty,E']}\big] \;\le\; \delta_{\mathrm{exact}}(-E')\,. \tag{5.2a}$$

(ii′) *For every $E' < 0$ with $e' := -\rho E'/\sqrt{2(d-1)} \ge 3/2$,*

$$\big(1 - 4\,\kappa(e')^{d-1}\big)\,\delta_{\mathrm{exact}}(-E') \;\le\; \mathbb{E}\big[N^{\mathrm{lm,min}}_{(-\infty,E']}\big] \;\le\; \delta_{\mathrm{exact}}(-E')\,. \tag{5.2b}$$

Proof. The Gaussian symmetry $X \overset{d}{=} -X$ gives $N^{\mathrm{lm,min}}_{(-\infty,E']} \overset{d}{=} N^{\mathrm{lm}}_{[-E',\infty)}$ for every $E' \in \mathbb{R}$, and Theorem 5.1 applies to the right-hand side. □

The bracket (ii′) is informative when $4\kappa(e')^{d-1} < 1$, that is for minima deeper than about twice the spectral edge ($e' \ge 9/4$ at every $d \ge 3$, and every fixed $e' > e_*$ for d large). The band $1 \le e' \le e_*$ is covered by (i′) only; in the limit $d \to \infty$ it contains the reduced ground-state energy E_0/E_∞ of Auffinger et al. [2, Eq. (2.22) and Theorem 2.12] (1.015 for $k = 3$, 1.13 for $k = 10$, below e_* for every $k \le 7 \cdot 10^4$).

Remark 5.3 (Exact Kac–Rice representation)**.** The Kac–Rice formula with the conditional law (1.1b) of the Hessian yields, for every $E \in \mathbb{R}$,

$$\mathbb{E}\big[N^{\mathrm{lm}}_{[E,\infty)}\big] \;=\; C_{k,d} \int_E^\infty \mathbb{E}\big[|\det(G - \rho x\, I_{d-1})|\, \mathbf{1}\{G - \rho x\, I_{d-1} \prec 0\}\big]\, \varphi(x)\, \mathrm{d}x\,, \tag{5.3}$$

with $C_{k,d}$ as in (1.1c); dropping the indicator gives (5.1b).

Corollary 5.4 (Asymptotic match with Auffinger et al. [2] in the high-energy regime)**.** *For $u \le -E_\infty$,*

$$\Theta_k(u) \;=\; \tfrac{1}{2}\log(k-1) \;-\; \tfrac{k-2}{4(k-1)}u^2 \;-\; I_1(u)\,, \qquad I_1(u) := \tfrac{2}{E_\infty^2}\int_u^{-E_\infty} \sqrt{z^2 - E_\infty^2}\,\mathrm{d}z\,. \tag{5.4a}$$

As $d \to \infty$ with $E = e\sqrt{2(d-1)}/\rho$ for fixed reduced energy $e \ge 1$,

$$\lim_{d\to\infty} \frac{1}{d} \log \mathbb{E}\big[N^{\mathrm{lm}}_{[E,\infty)}\big] \;=\; \Theta_k(-eE_\infty)\,, \tag{5.4b}$$

and

$$\Theta_k(-eE_\infty) \;=\; \tfrac{1}{2}\log(k-1) \;-\; \tfrac{(k-2)e^2}{k} \;-\; 2\int_1^e \sqrt{w^2-1}\,\mathrm{d}w\,. \tag{5.4c}$$

Proof. The function Θ_k is the complexity function of Auffinger et al. [2, Eqs. (2.14) and (2.16), Theorem 2.5] at $p = k$; (5.4c) is (5.4a) after the substitution $z = -E_\infty w$ in I_1. The limit (5.4b) follows from the equality $\mathbb{E}[N^{\mathrm{cp}}_{[E,\infty)}] = \delta_{\mathrm{exact}}(E)$ of part (i) of Theorem 5.1, the level correspondence $E/\sqrt{d} \to eE_\infty$, and Auffinger et al. [2, Theorems 2.5 and 2.8]: Step 3 of the proof of Theorem 5.1 in Appendix A. □

Exponentially many local maxima persist throughout the band $E_\infty < E/\sqrt{d} < E_0$, where $\Theta_k(-eE_\infty) > 0$. The restriction $e \ge 3/2$ concerns only the non-asymptotic bracket (5.1c), which is informative when $4\kappa(e)^{d-1} < 1$. The limit (5.4b) holds on the whole range $e \ge 1$, since it follows from part (i) and Auffinger et al. [2, Theorems 2.5 and 2.8]. A finite-d lower bracket on $1 < e \le e_*$ would need the large deviations of $\lambda_{\max}(G)$ near the edge [6, Theorem 6.2] in place of the Gaussian concentration of Lemma 4.9(ii); we leave it open.

Remark 5.5 (Normalization reconciliation with Auffinger et al. [2])**.** Auffinger et al. [2] use $H_{N,p}$ on the sphere of radius $\sqrt{N}$ ($\mathbb{E}[H^2] = N$) with energy $u = H/N$, so that our level E is their $u_{\mathrm{ABC}} = E/\sqrt{d}$ with ambient dimension $N = d$ (Step 3b of the proof in Appendix A); above the spectral edge E_∞, critical points are local maxima with overwhelming probability, E_∞ being the right edge of the limiting GOE spectral support; we use $f = H(\sqrt{N}\,\cdot)/\sqrt{N}$ on the unit sphere ($\mathbb{E}[f^2] = 1$) with energy E on the scale $\sqrt{N}\,u$. By (A.10), $e = \sqrt{d/(d-1)}\,u_{\mathrm{ABC}}/E_\infty$, so the two conventions agree as $d \to \infty$.

Remark 5.6 (Three nested non-asymptotic complexity bounds)**.** Substituting δ_{IMF} (for $E \ge u_{\mathrm{IMF}}$) or δ_{SMF} (for $E \ge u_{\mathrm{SMF}}$) for δ_{exact} in (5.1c) yields three nested non-asymptotic upper bounds on the annealed complexity: δ_{exact} tightest (Theorem 2.4), δ_{IMF} asymptotically sharp (Theorem 2.7), δ_{SMF} a single monomial past $u_d^\star$ (Corollary 2.11); the lower bound of (5.1c) is unchanged.

A Deferred proofs

For readability the three longest proofs are collected here.

Proof of Theorem 2.4. We check each D_i in turn; only D_2 requires care.

• *D_4 (squared Hermite).* By Lemma 2.2 the squared-Hermite part of $Q_d(\nu)$ is $e^{-\nu^2/2}\sum_{j=0}^{d-1} c_j^2 H_j(\nu)^2$, so with $\nu = \rho x$ and the change of variable $y = \rho x$,

$$\int_u^\infty H_j(\rho x)^2\, e^{-(1+\rho^2)x^2/2}\,\mathrm{d}x \;=\; \frac{1}{\rho}\int_{\rho u}^\infty H_j(y)^2\, e^{-\beta y^2/2}\,\mathrm{d}y \;=\; \frac{1}{\rho}\,K_j^\beta(\rho u)\,, \qquad \beta \;=\; \frac{1+\rho^2}{\rho^2} \;=\; \frac{3k-2}{k}\,,$$

and summing over j,

$$D_4(u) \;=\; T_d^{\mathrm{exact}}(u) \;:=\; \frac{1}{\rho}\sum_{j=0}^{d-1} c_j^2\, K_j^{\beta}(\rho u)\,, \tag{A.1}$$

each K_j^{β} being given in closed form by Corollary B.5; no inequality is used in this step.

• D_1, D_3 *(linear Hermite).* The change of variable $y = \rho x$ in $\mathcal{L}_d(u) = \int_u^\infty H_{d-1}(\rho x)\, e^{-x^2/2}\,\mathrm{d}x$ gives $\mathcal{L}_d(u) = \rho^{-1} J_{d-1}^{1/\rho^2}(\rho u)$, and Lemma B.4 with $\beta = 1/\rho^2$, $\gamma = 2\rho^2$, $\vartheta = 2\rho^2 - 1 = \Lambda$ supplies the closed form. For d odd, D_3 exhibits a Mills-ratio-type remainder $\bar\Phi(\rho u\sqrt{\beta}) = \bar\Phi(u)$ (since $d-1$ is even and J_0^{1/ρ^2} enters); for d even, D_1 has no such remainder (since $d-1$ is odd).

• D_2*, cross-integral, polynomial part.* Apply Lemma B.3 to $\mathcal{I}_d^c(\rho x)$:

$$\mathcal{I}_d^c(\rho x) \;=\; e^{-(\rho x)^2/2}\sum_{\ell=0}^{\lfloor (d-1)/2\rfloor} \frac{2^{\ell+1}(d-1)!!}{(d-2\ell-1)!!}\, H_{d-2\ell-1}(\rho x) \;+\; \mathbf{1}\{d\,\text{even}\}\, 2^{d/2}(d-1)!! \int_{\rho x}^\infty e^{-y^2/2}\,\mathrm{d}y\,.$$

Substituting into (2.5b), the Hermite-series part becomes

$$\sum_{\ell} \frac{2^{\ell+1}(d-1)!!}{(d-2\ell-1)!!}\int_u^\infty H_{d-1}(\rho x)\, H_{d-2\ell-1}(\rho x)\, e^{-(1+\rho^2)x^2/2}\,\mathrm{d}x\,.$$

The classical product linearization $H_a(\rho x)\, H_b(\rho x) = \sum_{p=0}^{\min(a,b)} 2^p\, p!\, \binom{a}{p}\binom{b}{p}\, H_{a+b-2p}(\rho x)$, combined with $y = \rho x$ and Lemma B.4 with $\beta = (3k-2)/k$, evaluates each integral to $\rho^{-1} J_{a+b-2p}^{\beta}(\rho u)$, producing the first line of (2.6c).

• D_2*, cross-integral, Fubini remainder (d even only).* The remainder contribution to $\mathcal{C}_d(u)$ is, after substituting $\mathcal{I}_d^c$'s Gaussian-tail piece,

$$2^{d/2}(d-1)!!\,\mathcal{F}_d(u)\,, \qquad \mathcal{F}_d(u) \;:=\; \int_u^\infty H_{d-1}(\rho x)\left[\int_{\rho x}^\infty e^{-y^2/2}\,\mathrm{d}y\right] e^{-x^2/2}\,\mathrm{d}x\,.$$

The double integral is over $\{(x,y) : x \ge u,\ y \ge \rho x\}$. By Fubini, swapping the order, y ranges over $[\rho u, \infty)$ and for each such y, x ranges over $[u, y/\rho]$:

$$\mathcal{F}_d(u) \;=\; \int_{\rho u}^\infty e^{-y^2/2}\int_u^{y/\rho} H_{d-1}(\rho x)\, e^{-x^2/2}\,\mathrm{d}x\,\mathrm{d}y \;=\; \frac{1}{\rho}\int_{\rho u}^\infty e^{-y^2/2}\left[J_{d-1}^{1/\rho^2}(\rho u) - J_{d-1}^{1/\rho^2}(y)\right]\mathrm{d}y\,,$$

where the second equality applies the change of variable $z = \rho x$ to the inner integral and writes the difference of J_{d-1}^{1/ρ^2} at the two endpoints. Splitting at the bracket gives

$$\mathcal{F}_d(u) \;=\; \underbrace{\frac{J_{d-1}^{1/\rho^2}(\rho u)}{\rho}\int_{\rho u}^\infty e^{-y^2/2}\,\mathrm{d}y}_{=:\,\mathcal{F}_d^{(1)}(u)} \;-\; \underbrace{\frac{1}{\rho}\int_{\rho u}^\infty e^{-y^2/2}\, J_{d-1}^{1/\rho^2}(y)\,\mathrm{d}y}_{=:\,\mathcal{F}_d^{(2)}(u)}\,. \tag{A.2}$$

First piece $\mathcal{F}_d^{(1)}$. Direct evaluation: $\int_{\rho u}^\infty e^{-y^2/2}\,\mathrm{d}y = \sqrt{2\pi}\,\bar\Phi(\rho u)$, so

$$\mathcal{F}_d^{(1)}(u) \;=\; \sqrt{2\pi}\,\bar\Phi(\rho u)\,\frac{J_{d-1}^{1/\rho^2}(\rho u)}{\rho}\,,$$

which is the first term of (2.6d).

Second piece $\mathcal{F}_d^{(2)}$. For d even, $d-1$ is odd, so the iterated form (B.5d) of Lemma B.4 applied with $\beta = 1/\rho^2$, $\gamma = 2/\beta = 2\rho^2$, $\vartheta = (2-\beta)/\beta = 2\rho^2 - 1 = \Lambda$ has $R_{d-1}^{1/\rho^2} = 0$ and reads

$$J_{d-1}^{1/\rho^2}(y) \;=\; 2\rho^2\, e^{-y^2/(2\rho^2)} \sum_{\ell=0}^{(d-2)/2} (2\Lambda)^\ell \, \frac{(d-2)!!}{(d-2\ell-2)!!}\, H_{d-2\ell-2}(y)\,.$$

Substituting into $\mathcal{F}_d^{(2)}(u)$ and pulling the constants $2\rho^2$ and the outer $1/\rho$ outside the sum,

$$\begin{aligned}
\mathcal{F}_d^{(2)}(u) \;&=\; \frac{1}{\rho}\int_{\rho u}^{\infty} e^{-y^2/2}\cdot 2\rho^2\, e^{-y^2/(2\rho^2)} \sum_{\ell=0}^{(d-2)/2} (2\Lambda)^\ell \, \frac{(d-2)!!}{(d-2\ell-2)!!}\, H_{d-2\ell-2}(y)\,\mathrm{d}y \\
&=\; \underbrace{\frac{2\rho^2}{\rho}}_{=2\rho} \sum_{\ell=0}^{(d-2)/2} (2\Lambda)^\ell \, \frac{(d-2)!!}{(d-2\ell-2)!!} \int_{\rho u}^{\infty} H_{d-2\ell-2}(y)\, e^{-(1+1/\rho^2)y^2/2}\,\mathrm{d}y \\
&=\; 2\rho \sum_{\ell=0}^{(d-2)/2} (2\Lambda)^\ell \, \frac{(d-2)!!}{(d-2\ell-2)!!}\, J_{d-2\ell-2}^{\beta}(\rho u)\,,
\end{aligned}$$

where the combined exponent $-(1/\rho^2+1)y^2/2 = -\beta y^2/2$ identifies $\beta = 1 + 1/\rho^2 = (1+\rho^2)/\rho^2 = (3k-2)/k$, the same β appearing in D_4 (Corollary B.5), and the last line applies the definition $J_m^\beta(\rho u) = \int_{\rho u}^\infty H_m(y)\, e^{-\beta y^2/2}\,\mathrm{d}y$.

Assembly. Substituting $\mathcal{F}_d^{(1)}$ and $\mathcal{F}_d^{(2)}$ into (A.2) and noting the explicit minus sign in front of $\mathcal{F}_d^{(2)}$, the second-piece coefficient becomes -2ρ, yielding the second line of (2.6d).

• *Pointwise tightness.* Let $u \ge u_{\mathrm{IMF}}$. On $\{\rho x \ge \sqrt{2d-1}\}$ both $\mathcal{I}_d^c(\rho x) > 0$ (Lemma 2.5) and $H_{d-1}(\rho x) > 0$, so $D_2(u) < 0$; with $D_1 + D_3 = \alpha_d\, \mathcal{L}_d(u) \le \alpha_d\, \Phi_d(\rho, u)\, e^{-u^2/2}$ (Proposition 2.6) this gives $\delta_{\mathrm{exact}}(u) < \delta_{\mathrm{IMF}}^\star(u)$. For the second inequality, $\sqrt{2d-1} > x_1^{(j)}$ for every $j \le d-1$, so Lemma 2.8 gives $0 \le H_j(\rho x) \le (2\rho x)^j$ on $\{\rho x \ge \sqrt{2d-1}\}$, with equality for $j = 0, 1$ and strict inequality for $j \ge 2$; hence

$$\int_u^\infty H_j(\rho x)^2\, e^{-(1+\rho^2)x^2/2}\,\mathrm{d}x \;\le\; \int_u^\infty (2\rho x)^{2j}\, e^{-(1+\rho^2)x^2/2}\,\mathrm{d}x\,,$$

strictly for $j \ge 2$, an index present since $d \ge 3$. Summing with the weights c_j^2 and bounding the right-hand side as in Proposition 2.6 gives $D_4(u) < \Psi_d(\rho, u)\, e^{-(1+\rho^2)u^2/2}$, that is $\delta_{\mathrm{IMF}}^\star(u) < \delta_{\mathrm{IMF}}(u)$. □

Proof of Lemma 2.9. • We separate the Mehta expansion (2.2) into three distinct pieces, $Q_d = T_1 + T_2 + T_3$, defined as follows:

$$\begin{aligned}
T_1(\nu) &= e^{-\nu^2/2} \sum_{j=0}^{d-1} c_j^2 H_j^2(\nu)\,, \\
T_2(\nu) &= \tfrac{1}{2}\sqrt{d/2}\, c_{d-1} c_d\, H_{d-1}(\nu)\,\big[\mu_d - 2\mathcal{I}_d^c(\nu)\big]\,, \\
T_3(\nu) &= \mathbf{1}\{d \text{ odd}\}\, \frac{H_{d-1}(\nu)}{\mu_{d-1}}\,.
\end{aligned}$$

• We extract the constant prefactor multiplying $H_{d-1}(\nu)$, which depends on the parity of the dimension d:

- **For d even:** $T_3 = 0$, and the constant part of the bracket in T_2 isolates $\alpha_d = \tfrac{1}{2}\sqrt{d/2}\, c_{d-1} c_d\, \mu_d$.

- **For d odd:** $\mu_d = 0$ (by parity), leaving only T_3 to contribute. This gives $\alpha_d = 1/\mu_{d-1}$ (here $d-1$ is even, so $\mu_{d-1} \neq 0$).

The exact values of the integrals are obtained via direct evaluation of the Gaussian moments

$$\int_{\mathbb{R}} y^{2j}\, e^{-y^2/2}\, \mathrm{d}y \;=\; \sqrt{2\pi}\,(2j-1)!!\,,$$

yielding the closed form $\mu_{2p} = \sqrt{2\pi}\,(2p)!/p!$.
• The total remainder $\mathcal{R}_d(\nu) := Q_d(\nu) - \alpha_d H_{d-1}(\nu)$ has the same closed form in both parities:

- **For d even**, $T_3 = 0$ and the constant part of T_2's bracket is $\alpha_d H_{d-1}(\nu)$, so $\mathcal{R}_d = T_1 + (T_2 - \alpha_d H_{d-1}) = T_1 - \sqrt{d/2}\, c_{d-1} c_d\, \mathcal{I}_d^c(\nu)\, H_{d-1}(\nu)$.
- **For d odd**, $\mu_d = 0$ so $T_2 = -\sqrt{d/2}\, c_{d-1} c_d\, \mathcal{I}_d^c(\nu)\, H_{d-1}(\nu)$, and $T_3 = H_{d-1}(\nu)/\mu_{d-1} = \alpha_d H_{d-1}(\nu)$, so $\mathcal{R}_d = T_1 + T_2 = T_1 - \sqrt{d/2}\, c_{d-1} c_d\, \mathcal{I}_d^c(\nu)\, H_{d-1}(\nu)$.

In both cases

$$\mathcal{R}_d(\nu) \;=\; T_1(\nu) \;-\; \sqrt{d/2}\, c_{d-1} c_d\, \mathcal{I}_d^c(\nu)\, H_{d-1}(\nu)\,, \tag{A.3}$$

and we bound the two terms separately.

We first bound T_1. The physicist Hermite polynomial admits the explicit expansion

$$H_j(\nu) \;=\; \sum_{m=0}^{\lfloor j/2 \rfloor} \frac{(-1)^m\, j!}{m!\,(j-2m)!} (2\nu)^{j-2m}\,,$$

and we use the elementary coefficient inequality $j!\, 2^{j-2m}/m! \le (2j)!/(2m)!$, valid for every $j \ge m \ge 0$. The latter is proved by induction: the base case $j = m$ gives $2^{-m} \le 1$, and the step $j \to j+1$ multiplies the LHS by $2(j+1)$ and the RHS by $2(2j+1)(j+1)$, giving a ratio of $2j+1 \ge 1$. The triangle inequality then yields

$$|H_j(\nu)| \;\le\; \sum_{m=0}^{\lfloor j/2 \rfloor} \frac{(2j)!}{(2m)!(j-2m)!} |\nu|^{j-2m} \;\le\; \sum_{p=0}^{j} \frac{(2j)!}{j!} \binom{j}{p} |\nu|^p \;=\; \frac{(2j)!}{j!} (1+|\nu|)^j\,,$$

where the second inequality adds the (non-negative) odd-power monomials $|\nu|^p$ with $p \not\equiv j \pmod 2$, using $(2j)!/((2m)!(j-2m)!) = (2j)!/j! \cdot j!/((2m)!(j-2m)!) \le (2j)!/j! \cdot \binom{j}{j-2m}$. Squaring and applying $(1+|\nu|)^2 \le 2(1+\nu^2)$,

$$T_1(\nu) \;\le\; S_d (1+\nu^2)^{d-1} e^{-\nu^2/2}\,, \qquad \text{with } S_d \text{ as in Lemma 2.9.}$$

• *A uniform envelope for $\mathcal{I}_d^c$.* We bound $\mathcal{I}_d^c$ on the whole half-line $\{\nu \ge 0\}$ by a single envelope carrying the same polynomial–Gaussian form $(1+|\nu|)^{d-1} e^{-\nu^2/2}$ as H_{d-1}. The explicit Hermite-tail series of Lemma B.3 (applied with degree $m = d$) reads

$$\mathcal{I}_d^c(\nu) \;=\; e^{-\nu^2/2} \sum_{\ell=0}^{\lfloor (d-1)/2 \rfloor} 2^{\ell+1} \frac{(d-1)!!}{(d-2\ell-1)!!}\, H_{d-2\ell-1}(\nu) \;+\; R_d(\nu)\,,$$

with $R_d(\nu) = 0$ for d odd and $R_d(\nu) = 2^{d/2}(d-1)!! \int_\nu^\infty e^{-y^2/2}\, \mathrm{d}y$ for d even. By the coefficient bound $|H_m(\nu)| \le \frac{(2m)!}{m!}(1+|\nu|)^m$ established above for T_1 (whence $|H_{d-2\ell-1}(\nu)| \le \frac{(2(d-2\ell-1))!}{(d-2\ell-1)!}(1+|\nu|)^{d-2\ell-1} \le \frac{(2(d-2\ell-1))!}{(d-2\ell-1)!}(1+|\nu|)^{d-1}$ since $1+|\nu| \ge 1$) and the monotone-Mills bound $\int_\nu^\infty e^{-y^2/2}\, \mathrm{d}y \le \sqrt{\pi/2}\, e^{-\nu^2/2}$ (valid

for every $\nu \geq 0$, because $\nu \mapsto e^{\nu^2/2}\int_\nu^\infty e^{-y^2/2}\,\mathrm{d}y$ is non-increasing, with value $\sqrt{\pi/2}$ at $\nu = 0$, since its derivative $\nu\, e^{\nu^2/2}\int_\nu^\infty e^{-y^2/2}\mathrm{d}y - 1$ is negative by the Mills inequality $\nu\, e^{\nu^2/2}\int_\nu^\infty e^{-y^2/2}\mathrm{d}y < 1$), we obtain

$$\begin{aligned} |\mathcal{I}_d^c(\nu)| \;&\leq\; E_d\,(1+|\nu|)^{d-1}\,e^{-\nu^2/2}\,, \\ E_d \;&:=\; \sum_{\ell=0}^{\lfloor (d-1)/2\rfloor} 2^{\ell+1}\frac{(d-1)!!}{(d-2\ell-1)!!}\frac{(2(d-2\ell-1))!}{(d-2\ell-1)!} \;+\; 2^{d/2}(d-1)!!\sqrt{\tfrac{\pi}{2}}\,. \end{aligned} \tag{A.4}$$

• *E_d is dominated by $\widetilde{B}_d$.* The summands $a_\ell := 2^{\ell+1}\frac{(d-1)!!}{(d-2\ell-1)!!}\frac{(2(d-2\ell-1))!}{(d-2\ell-1)!}$ are strictly decreasing in ℓ: with $p := d-2\ell-1 \geq 2$, a direct computation gives the ratio $a_{\ell+1}/a_\ell = \frac{p}{2(2p-1)(2p-3)} \leq \frac{1}{2(2p-3)} < 1$. Hence, using $a_0 = 2\frac{(2d-2)!}{(d-1)!}$ and that the sum has at most $\lfloor (d-1)/2\rfloor + 1 \leq \frac{d+1}{2}$ terms,

$$\sum_\ell a_\ell \;\leq\; \frac{d+1}{2}\,a_0 \;=\; (d+1)\frac{(2d-2)!}{(d-1)!}\,, \qquad 2^{d/2}(d-1)!!\sqrt{\tfrac{\pi}{2}} \;\leq\; 2^d\frac{(2d-2)!}{(d-1)!}\,,$$

the second bound using $(d-1)!! \leq (d-1)! \leq \frac{(2d-2)!}{(d-1)!}$ and $\sqrt{\pi/2} \leq 2^{d/2}$. Therefore

$$E_d \;\leq\; (d+1+2^d)\frac{(2d-2)!}{(d-1)!} \;\leq\; (2d-1)\,2^{d+2}\,\frac{(2d-2)!}{(d-1)!} \;=\; \frac{(2d)!\,2^{d+1}}{d!} \;=\; \widetilde{B}_d\,,$$

so that, by (A.4), $|\mathcal{I}_d^c(\nu)| \leq \widetilde{B}_d\,(1+|\nu|)^{d-1}\,e^{-\nu^2/2}$ for every $\nu \geq 0$.

• *Assembly.* Combining the last bound with $|H_{d-1}(\nu)| \leq \frac{(2d-2)!}{(d-1)!}(1+|\nu|)^{d-1}$ and the elementary inequality $(1+|\nu|)^{2(d-1)} \leq 2^{d-1}(1+\nu^2)^{d-1}$ (which follows from $(1+|\nu|)^2 \leq 2(1+\nu^2)$), the cross term in (A.3) obeys

$$\sqrt{d/2}\,c_{d-1}c_d\,|\mathcal{I}_d^c(\nu)|\,|H_{d-1}(\nu)| \;\leq\; \sqrt{d/2}\,c_{d-1}c_d\,\frac{(2d-2)!}{(d-1)!}\,2^{d-1}\,\widetilde{B}_d\,(1+\nu^2)^{d-1}e^{-\nu^2/2}\,.$$

Adding the bound $T_1(\nu) \leq S_d\,(1+\nu^2)^{d-1}e^{-\nu^2/2}$ established above yields the remainder bound (2.10d) with the constant β_d of (2.10a). □

Proof of Theorem 5.1 and Corollary 5.4. The inputs are the Kac–Rice formula of Section 1.1, the conditional GOE law of the Hessian [2, Lemma 3.2(b)], and the two facts on the GOE of Lemma 4.9.

• *Step 1: Exact Kac–Rice representation and upper bound.* The Kac–Rice computation is the one of Section 1.1, displays (1.1b)–(1.1c), with one change: the target is the expected count of local maxima with value in $[E,\infty)$, for which the Kac–Rice formula of Section 1.1 is an equality, so the indicator $\mathbf{1}\{\nabla^2 X(t) \prec 0\}$ is kept inside the conditional expectation. By (1.1b), $\nabla^2 X(t) \prec 0$ if and only if $G - \rho x\, I_{d-1} \prec 0$, that is $\lambda_{\max}(G) < \rho x$. The constants gather as in (1.1c) and give the exact representation (5.3). Dropping the indicator $\mathbf{1}\{G - \rho x\, I_{d-1} \prec 0\} \leq 1$ gives $\mathbb{E}[N^{\mathrm{lm}}_{[E,\infty)}] \leq C_{k,d}\int_E^\infty A(\rho x)\,\varphi(x)\,\mathrm{d}x = \delta_{\mathrm{exact}}(E)$ by (1.1c), which proves part (i).

• *Step 2: Lower bound for $e \geq 3/2$.* Write $N := d-1$, $\lambda_{\max} := \lambda_{\max}(G)$, and fix $x \geq E$; put $\nu := \rho x$ and $e_x := \nu/\sqrt{2N} \geq e \geq 3/2$. Since $G - \nu I \prec 0$ if and only if $\lambda_{\max} < \nu$, and $|\det(G-\nu I)| = \det(\nu I - G)$ on that event,

$$\mathbb{E}\big[|\det(G-\nu I)|\,\mathbf{1}\{G-\nu I \prec 0\}\big] \;=\; A(\nu) \;-\; \mathbb{E}\big[|\det(\nu I - G)|\,\mathbf{1}\{\lambda_{\max} \geq \nu\}\big]\,, \qquad A(\nu) = \mathbb{E}|\det(G-\nu I)|\,. \tag{A.5}$$

(2a) Lower bound on $A(\nu)$. By Lemma 4.9(i), $\mathbb{E}\det(\nu I - G) = \sum_i \frac{(-1)^i N!}{i!\,(N-2i)!\,4^i}\nu^{N-2i} = 2^{-N}H_N(\nu)$, the physicists' Hermite polynomial, whose zeros $x_1 > x_2 > \cdots > x_N$ are symmetric about 0 and satisfy $x_1 < \sqrt{2N+1}$ (Szegő, as in Lemma 2.5). Since $\nu \ge \frac{3}{2}\sqrt{2N} > \sqrt{2N+1}$, pairing the zeros $\pm x_j$ (for N odd the zero at 0 contributes a factor $\nu \ge (\nu^2 - x_1^2)^{1/2}$),

$$A(\nu) \;\ge\; \mathbb{E}\det(\nu I - G) \;=\; \prod_{j=1}^{N}(\nu - x_j) \;\ge\; (\nu^2 - x_1^2)^{N/2} \;\ge\; (\nu^2 - 2N - 1)^{N/2}\,. \tag{A.6}$$

(2b) Upper bound on the remainder. Since $|\det(\nu I - G)| \le (\nu + \|G\|)^N$ and $\{\lambda_{\max} \ge \nu\} \subset \{\|G\| \ge \nu\}$, the layer-cake formula for $s \mapsto (\nu+s)^N$ gives

$$\mathbb{E}\big[|\det(\nu I - G)|\,\mathbf{1}\{\lambda_{\max} \ge \nu\}\big] \;\le\; (2\nu)^N\,\mathbb{P}\{\|G\| \ge \nu\} \;+\; N\int_{\nu}^{\infty}(\nu+s)^{N-1}\,\mathbb{P}\{\|G\| \ge s\}\,\mathrm{d}s\,.$$

By Lemma 4.9(ii), with $t_0 := \nu - \sqrt{2N} = (e_x - 1)\sqrt{2N}$ and the substitution $s = \sqrt{2N} + t$,

$$\mathbb{E}\big[|\det(\nu I - G)|\,\mathbf{1}\{\lambda_{\max} \ge \nu\}\big] \;\le\; 2(2\nu)^N\exp(-t_0^2/2) \;+\; 2N\int_{t_0}^{\infty}(\nu + \sqrt{2N} + t)^{N-1}\exp(-t^2/2)\,\mathrm{d}t\,.$$

For $t \ge t_0$, $-\frac{\mathrm{d}}{\mathrm{d}t}\big[(\nu+\sqrt{2N}+t)^{N-1}\exp(-t^2/2)\big] = (\nu+\sqrt{2N}+t)^{N-1}\exp(-t^2/2)\big(t - \frac{N-1}{\nu+\sqrt{2N}+t}\big)$ and $t - \frac{N-1}{\nu+\sqrt{2N}+t} \ge t_0 - \frac{N}{2\nu} = \sqrt{2N}\big((e_x-1) - \frac{1}{4e_x}\big) =: c > 0$, so, as $\nu + \sqrt{2N} + t_0 = 2\nu$ and $2\nu c = N\big(4e_x(e_x-1) - 1\big) \ge 2N$ for $e_x \ge 3/2$,

$$\begin{aligned} N\int_{t_0}^{\infty}(\nu+\sqrt{2N}+t)^{N-1}\exp(-t^2/2)\,\mathrm{d}t \;&\le\; \frac{N(2\nu)^{N-1}\exp(-t_0^2/2)}{c} \\ &=\; \frac{(2\nu)^N\exp(-t_0^2/2)}{4e_x(e_x-1)-1} \;\le\; \tfrac{1}{2}(2\nu)^N\exp(-t_0^2/2)\,. \end{aligned}$$

Hence

$$\mathbb{E}\big[|\det(\nu I - G)|\,\mathbf{1}\{\lambda_{\max} \ge \nu\}\big] \;\le\; 3\,(2\nu)^N\,\exp\big(-(e_x-1)^2 N\big)\,. \tag{A.7}$$

(2c) The amplifier. Dividing (A.7) by (A.6) and writing $\nu = e_x\sqrt{2N}$,

$$\frac{\mathbb{E}\big[|\det(\nu I - G)|\,\mathbf{1}\{\lambda_{\max} \ge \nu\}\big]}{A(\nu)} \;\le\; 3\left[\frac{2e_x\exp(-(e_x-1)^2)}{\sqrt{e_x^2 - 1 - \frac{1}{2N}}}\right]^N \;\le\; 3\left[\frac{2e\,\exp(-(e-1)^2)}{\sqrt{e^2-1-\frac{1}{2N}}}\right]^N,$$

the bracket being decreasing in e_x on $[3/2,\infty)$ (its logarithmic derivative is $\frac{1}{e_x} - 2(e_x-1) - \frac{e_x}{e_x^2-1-1/(2N)}$, and $\frac{1}{e_x} - 2(e_x-1) \le -\frac{1}{3}$ there). Finally $e^2 - 1 \ge \frac{5}{4}$ and Bernoulli's inequality give $\big(1 - \frac{1}{2N(e^2-1)}\big)^{-N/2} \le \big(1 - \frac{1}{4(e^2-1)}\big)^{-1} \le \frac{5}{4}$, whence

$$\frac{\mathbb{E}\big[|\det(\nu I - G)|\,\mathbf{1}\{\lambda_{\max} \ge \nu\}\big]}{A(\nu)} \;\le\; \tfrac{15}{4}\,\kappa(e)^N \;\le\; 4\,\kappa(e)^{d-1} \qquad \text{for every } x \ge E\,. \tag{A.8}$$

(2d) Integration. By (A.5) and (A.8), $\mathbb{E}[|\det(G - \rho x I)|\mathbf{1}\{G - \rho x I \prec 0\}] \ge (1 - 4\kappa(e)^{d-1})A(\rho x)$ for all $x \ge E$; multiplying by $C_{k,d}\varphi(x)$, integrating over $[E,\infty)$ and using (5.3) together with $C_{k,d}\int_E^\infty A(\rho x)\varphi(x)\,\mathrm{d}x = \delta_{\text{exact}}(E)$ from Step 1 gives the lower bound in (5.1c).

• *Step 3: High-dimensional limit via the ABC complexity theorem.* We avoid redoing the asymptotic of δ_{exact} from scratch via the Mehta–Fyodorov decomposition, which would require fine control of the partial Hermite sum $\Phi_d(\rho, E)$ at the natural scale $\rho E = e\sqrt{2(d-1)} \asymp \sqrt{d}$, where the leading-monomial approximation $\Phi_d \sim 2\rho(2\rho E)^{d-2}$ is invalid (every term in the sum contributes at comparable order). Instead,

we reduce (5.4b) directly to Auffinger et al. [2, Theorem 2.8] via an explicit field/level correspondence between our paper's KSS field X on $\mathbb{S}^{d-1}$ and ABC's spherical k-spin Hamiltonian $H_{N,p}$ on $S^{N-1}(\sqrt{N})$.

(3a) Field and Hessian correspondence. Auffinger et al. [2] work, throughout their Section 3, with the rescaled field

$$f_{N,p}(s) \;:=\; \frac{1}{\sqrt{N}}\, H_{N,p}(\sqrt{N}\, s)\,, \qquad s \in \mathbb{S}^{N-1}\,, \tag{A.9a}$$

of unit variance on the unit sphere, with covariance $\mathbb{E}[f(s)f(s')] = \langle s, s'\rangle^p$ [2, eqs. (3.1)–(3.2)]. Identifying their N with d and p with k, the law of f on $\mathbb{S}^{d-1}$ coincides with that of our paper's X. ABC's conditional-Hessian formula [2, Lemma 3.2, part (b)] is stated directly in this unit-sphere setting: for every $s \in \mathbb{S}^{N-1}$ and every $x \in \mathbb{R}$,

$$\nabla^2 f(s) \,\big|\, f(s) = x \;\overset{d}{=}\; M^{N-1}\sqrt{2(N-1)\,p(p-1)} \;-\; x\,p\,I_{N-1}\,, \tag{A.9b}$$

where M^{N-1} is a GOE in the ABC normalization $\mathbb{E}[(M^{N-1}_{ij})^2] = (1+\delta_{ij})/(2(N-1))$. Setting $G_{N-1} := \sqrt{N-1}\,M^{N-1}$, which has the normalization $\mathbb{E}[(G_{N-1})^2_{ij}] = (1+\delta_{ij})/2$, and using $p/\sqrt{2p(p-1)} = \sqrt{p/(2(p-1))} =: \rho$, the right-hand side rearranges to $\sqrt{2p(p-1)}\,(G_{N-1} - \rho\, x\, I_{N-1})$. With $(N,p) = (d,k)$ this is exactly the conditional Hessian law (1.1b)

$$\nabla^2 X(s) \,\big|\, X(s) = x \;\overset{d}{=}\; \sqrt{2k(k-1)}\,(G - \rho\, x\, I_{d-1})\,, \qquad \rho = \sqrt{k/(2(k-1))}\,,$$

the law used throughout the paper. No metric rescaling of the Hessian under a sphere-radius change is needed: ABC perform the rescaling once in passing from $H_{N,p}$ on $S^{N-1}(\sqrt{N})$ to $f_{N,p}$ on $\mathbb{S}^{N-1}$ via (A.9a), and their Hessian formula (A.9b) is already on the unit sphere.

(3b) Level correspondence. The threshold $X(s) \geq E$ in our paper corresponds to $H(\sqrt{N}\,s) \geq E\sqrt{N}$, i.e., $H/N \geq E/\sqrt{N} =: u_{\mathrm{ABC}}$. On the scale $E = e\sqrt{2(d-1)}/\rho$, using $\rho E_\infty = \sqrt{2}$ (which follows from $\rho^2 = k/(2(k-1))$ and $E_\infty^2 = 4(k-1)/k$),

$$u_{\mathrm{ABC}} \;=\; \frac{E}{\sqrt{d}} \;=\; \frac{e\sqrt{2(d-1)}}{\rho\sqrt{d}} \;\xrightarrow[d\to\infty]{}\; \frac{e\sqrt{2}}{\rho} \;=\; \frac{e\sqrt{2}}{\rho E_\infty}\cdot E_\infty \;=\; e\,E_\infty\,. \tag{A.10}$$

For $e \geq 1$ the limiting level satisfies $-eE_\infty \leq -E_\infty$, so the case-1 branch of (5.4a) applies at eE_∞; at finite d, $u_{\mathrm{ABC}} = eE_\infty\sqrt{(d-1)/d} < eE_\infty$, and Step (3c) passes to the limit by monotonicity and continuity.

(3c) Asymptotic rate for the total critical-point count. By Auffinger et al. [2, Theorem 2.8] (log-rate for the total critical-point count below level Nu) combined with the upper-tail symmetry $H \overset{d}{=} -H$ (an immediate consequence of $J \overset{d}{=} -J$ in the disorder), for every fixed $u \geq E_\infty$ (for smaller u the rate is ABC's $\Theta_p(-u)$, whose formula differs from (5.4a) but which is continuous on $\mathbb{R}$):

$$\lim_{N\to\infty}\frac{1}{N}\log\mathbb{E}\big[\#\{\sigma : \nabla H(\sigma) = 0,\, H(\sigma) \in (Nu, \infty)\}\big] \;=\; \Theta_k(-u)\,.$$

Since $u_{\mathrm{ABC}} \to eE_\infty$, the count is non-increasing in the level and Θ_p is continuous, the limit persists along the d-dependent level u_{ABC}: bracket u_{ABC} between $eE_\infty \mp \eta$, apply the display at both fixed levels, and let $\eta \to 0$. By the Kac–Rice equality (1.1c) (recall from Step 1 that $\delta_{\mathrm{exact}}(E)$ equals the Kac–Rice expected critical-point count above level E), the left-hand side equals $\lim_d \frac{1}{d}\log\delta_{\mathrm{exact}}(E)$ under the field/level correspondence of (3a)–(3b). Hence

$$\lim_{d\to\infty}\frac{1}{d}\log\delta_{\mathrm{exact}}(E) \;=\; \Theta_k(-eE_\infty) \;=\; \tfrac{1}{2}\log(k-1) - \tfrac{(k-2)e^2}{k} - 2\int_1^e \sqrt{w^2-1}\,\mathrm{d}w \tag{A.11}$$

for $e \geq 1$, using (5.4c).

(3d) Local-maxima specialization and the bracketing. ABC's Theorem 2.8 covers the total critical-point count; for local maxima (Morse index $N-1$ on the sphere) specifically, Remark 2.6 following Auffinger

et al. [2, Theorem 2.5] gives $\lim_N \frac{1}{N}\log \mathbb{E}[\#\{\text{local maxima of } H \text{ above } Nu\}] = \Theta_{0,p}(-u)$, and the case-1 branch $\Theta_{0,p}(u) = \Theta_p(u)$ on $u \le -E_\infty$ [2, Eqs. (2.15) and (2.16)] states that for $e \ge 1$, the local-maxima rate coincides with the total-critical-point rate: every critical point above the spectral edge is, with overwhelming probability, a local maximum. Therefore, by the bracketing of (3c), $\Theta_{0,p}$ being continuous on $\mathbb{R}$,

$$\lim_{d\to\infty} \frac{1}{d}\log \mathbb{E}[N^{\mathrm{lm}}_{[E,\infty)}] \;=\; \Theta_k(-eE_\infty).$$

Equivalently, and consistently with our non-asymptotic bracketing of (5.1c), combining the upper bound $\mathbb{E}[N^{\mathrm{lm}}] \le \delta_{\mathrm{exact}}$ with the lower bound $\mathbb{E}[N^{\mathrm{lm}}] \ge (1 - 4\kappa(e)^{d-1})\delta_{\mathrm{exact}}$ (valid for $e \ge 3/2$), and observing that the amplifier $4\kappa(e)^{d-1}$ vanishes exponentially as $d \to \infty$ for $e > e_*$,

$$\tfrac{1}{d}\log \delta_{\mathrm{exact}} + \tfrac{1}{d}\log\left(1 - 4\kappa(e)^{d-1}\right) \;\le\; \tfrac{1}{d}\log \mathbb{E}[N^{\mathrm{lm}}] \;\le\; \tfrac{1}{d}\log \delta_{\mathrm{exact}}\,,$$

both bracketing terms converge to $\Theta_k(-eE_\infty)$ by (A.11). The limit (5.4b) is thus established for every $e \ge 1$ by the exact Kac–Rice equality of Step 1 and Auffinger et al. [2, Theorems 2.5 and 2.8] (Steps 3c–3d); on $e > e_*$ the non-asymptotic bracket above re-derives the same limit with explicit finite-d error bars. The band $1 \le e \le e_*$ is covered by the asymptotic statement (5.4b), while only the matching finite-d lower bracket, not the limit itself, lies outside the scope of the present argument. □

B Technical lemmas

B.1 Gaussian-type tail integrals

The proofs of the three tail bounds use a common tool: integration by parts on tail integrals of the form $\int_u^\infty x^m e^{-ax^2/2}\,\mathrm{d}x$, iterated in steps of two. We isolate it here.

Lemma B.1 (Gaussian-type tail integrals)**.** *For $a > 0$, integer $m \ge 0$, and $u > 0$, define $I_m(u;a) := \int_u^\infty x^m e^{-ax^2/2}\,\mathrm{d}x$. The base case $m = 0$ obeys the Mills-type bound*

$$I_0(u;a) \;=\; \int_u^\infty e^{-ax^2/2}\,\mathrm{d}x \;\le\; \frac{e^{-au^2/2}}{a\,u}\,, \tag{B.1a}$$

and $I_1(u;a) = a^{-1}e^{-au^2/2}$ exactly. For every integer $m \ge 2$,

$$I_m(u;a) \;=\; \frac{u^{m-1}\,e^{-au^2/2}}{a} \;+\; \frac{m-1}{a}\,I_{m-2}(u;a)\,, \tag{B.1b}$$

and, provided $(m-1)/(au^2) < 1$,

$$I_m(u;a) \;\le\; \frac{u^{m-1}\,e^{-au^2/2}}{a - (m-1)/u^2}\,. \tag{B.1c}$$

In particular,

$$I_{d-1}(u;1) \le 2u^{d-2}e^{-u^2/2}\,, \qquad \text{for } u^2 \ge 2(d-2)\,,$$
$$I_{2d-2}(u;3/2) \le u^{2d-3}e^{-3u^2/4}\,, \qquad \text{for } u \ge 2\sqrt{d}\,.$$

Proof. On $[u,\infty)$, $1 \le x/u$, so $I_0(u;a) \le u^{-1}\int_u^\infty x\,e^{-ax^2/2}\,\mathrm{d}x = e^{-au^2/2}/(au)$, which is (B.1a); I_1 is explicit. For $m \ge 2$, integrating $x^{m-1}\cdot x\,e^{-ax^2/2}$ by parts gives (B.1b). Iterating it down to I_1 or I_0, bounding I_0 by (B.1a) and each product $(m-1)(m-3)\cdots(m-2j+1)$ by $(m-1)^j$,

$$I_m(u;a) \;\le\; \frac{u^{m-1}e^{-au^2/2}}{a}\sum_{j\ge 0}\left(\frac{m-1}{au^2}\right)^j,$$

a geometric series whose sum is (B.1c) when $(m-1)/(au^2) < 1$. The two particular cases follow from (B.1c): in the respective ranges the denominators $1 - (d-2)/u^2$ and $\frac{3}{2} - (2d-3)/u^2$ are at least $\frac{1}{2}$ and 1. □

B.2 Asymptotic leading term

Lemma B.2 (Leading term)**.** *As* $\nu \to \infty$, $A(\nu) \sim \nu^{d-1}$; *consequently* $\int_u^\infty A(\rho x)\varphi(x)\,\mathrm{d}x \sim \int_u^\infty (\rho x)^{d-1}\varphi(x)\,\mathrm{d}x$ *as* $u \to \infty$.

Proof. For $\nu \geq 1$, $\nu^{-(d-1)}|\det(G-\nu I)| = \prod_i |1-\mu_i/\nu| \leq (1+\max_i|\mu_i|)^{d-1}$, which is integrable, and the product tends to 1 almost surely; dominated convergence gives the first claim. For the second, given $\varepsilon \in (0,1)$ pick x_ε with $|A(\rho x)/(\rho x)^{d-1} - 1| \leq \varepsilon$ for $x \geq x_\varepsilon$; for $u \geq x_\varepsilon$ the ratio of the two integrals lies in $[1-\varepsilon, 1+\varepsilon]$. Finally, (B.1b) with $a=1$, $m=d-1$ gives $I_{d-1}(u;1) \geq u^{d-2}e^{-u^2/2}$ and (B.1c) gives $I_{d-1}(u;1) \leq u^{d-2}e^{-u^2/2}/(1-(d-2)/u^2)$, so $\int_u^\infty x^{d-1}\varphi(x)\,\mathrm{d}x \sim u^{d-2}\varphi(u)$, and $C_{k,d}\,\rho^{d-1}/\sqrt{2\pi} = \sqrt{2}\,(k/2)^{(d-1)/2}/\Gamma(d/2)$ yields (1.2). □

B.3 Explicit Hermite tail integrals

Lemma B.3 (Explicit Hermite tail integrals)**.** *For every* $u \in \mathbb{R}$ *and integer* $m \geq 1$,

$$\int_u^\infty H_m(x)\,e^{-x^2/2}\,\mathrm{d}x \;=\; e^{-u^2/2}\sum_{\ell=0}^{\lfloor (m-1)/2\rfloor} 2^{\ell+1}\frac{(m-1)!!}{(m-2\ell-1)!!}H_{m-2\ell-1}(u) \;+\; R_m(u)\,, \tag{B.2}$$

where $R_m(u) = 0$ *if* m *is odd and* $R_m(u) = 2^{m/2}(m-1)!!\int_u^\infty e^{-x^2/2}\,\mathrm{d}x$ *if* m *is even.*

Proof. • We evaluate $J_m = \int_u^\infty H_m(x)e^{-x^2/2}\,\mathrm{d}x$. The integrations by parts below use only the finiteness of the boundary term $H_{m-1}(u)\,e^{-u^2/2}$ at u and the vanishing of the integrand at $+\infty$, so no positivity of u is required. We use two standard identities for the physicist Hermite polynomials:

$$H_m'(x) = 2mH_{m-1}(x) \tag{B.3a}$$
$$H_m(x) = 2xH_{m-1}(x) - 2(m-1)H_{m-2}(x) \tag{B.3b}$$

We begin by evaluating the integral of $2xH_{m-1}(x)e^{-x^2/2}$ using integration by parts. Let $f(x) = H_{m-1}(x)$ and $g'(x) = xe^{-x^2/2}$, which implies $g(x) = -e^{-x^2/2}$.

$$\begin{aligned}\int_u^\infty xH_{m-1}(x)e^{-x^2/2}\,\mathrm{d}x &= \Big[-H_{m-1}(x)e^{-x^2/2}\Big]_u^\infty + \int_u^\infty H_{m-1}'(x)e^{-x^2/2}\,\mathrm{d}x \\ &= H_{m-1}(u)e^{-u^2/2} + 2(m-1)\int_u^\infty H_{m-2}(x)e^{-x^2/2}\,\mathrm{d}x\end{aligned}$$

where we substituted (B.3a) in the second term. We then substitute $2xH_{m-1}(x)$ using a rearrangement of (B.3b), $2xH_{m-1}(x) = H_m(x) + 2(m-1)H_{m-2}(x)$, into the left-hand side:

$$\frac{1}{2}\int_u^\infty \Big(H_m(x) + 2(m-1)H_{m-2}(x)\Big)e^{-x^2/2}\,\mathrm{d}x = H_{m-1}(u)e^{-u^2/2} + 2(m-1)\int_u^\infty H_{m-2}(x)e^{-x^2/2}\,\mathrm{d}x$$

Distributing the integral and isolating $J_m = \int_u^\infty H_m(x)e^{-x^2/2}\,\mathrm{d}x$ yields the exact recurrence relation:

$$J_m = 2H_{m-1}(u)e^{-u^2/2} + 2(m-1)J_{m-2} \tag{B.4}$$

By iteratively applying (B.4), we can expand J_m:

$$\begin{aligned}J_m &= 2H_{m-1}(u)e^{-u^2/2} + 2(m-1)\Big[2H_{m-3}(u)e^{-u^2/2} + 2(m-3)J_{m-4}\Big] \\ &= e^{-u^2/2}\Big[2H_{m-1}(u) + 2^2(m-1)H_{m-3}(u) + 2^3(m-1)(m-3)H_{m-5}(u) + \dots\Big] + \text{Remainder}\end{aligned}$$

The ℓ-th term of this expansion involves the cascading product $(m-1)(m-3)\cdots(m-2\ell+1)$, which evaluates precisely to the double factorial ratio $\frac{(m-1)!!}{(m-2\ell-1)!!}$. Incorporating the accumulated factor of $2^{\ell+1}$, the summation becomes:

$$e^{-u^2/2}\sum_{\ell=0}^{\lfloor (m-1)/2\rfloor} 2^{\ell+1}\frac{(m-1)!!}{(m-2\ell-1)!!}H_{m-2\ell-1}(u)$$

The recursion stops differently depending on parity:

- If m is odd, the final evaluation step corresponds to $\ell=(m-1)/2$, meaning $H_0(u)$ is evaluated and no subsequent integrals are required. Thus, $R_m(u)=0$.
- If m is even, the final evaluated boundary term corresponds to $\ell=(m-2)/2$. The remaining integral falls back to $J_0=\int_u^\infty H_0(x)e^{-x^2/2}\,\mathrm{d}x=\int_u^\infty e^{-x^2/2}\,\mathrm{d}x$. The accumulated multiplier in front of J_0 is $2^{m/2}(m-1)!!$, yielding exactly the prescribed $R_m(u)$.

□

The next lemma extends Lemma B.3 from the canonical Gaussian weight $e^{-y^2/2}$ to an arbitrary weight $e^{-\beta y^2/2}$ with $\beta>0$. The general β-version gives the exact squared-Hermite tail $T_d^{\mathrm{exact}}(u)$ of the D_4 step in the proof of Theorem 2.4: when integrating $H_j(\rho x)^2$ against $e^{-(1+\rho^2)x^2/2}$ on $[u,\infty)$ and substituting $y=\rho x$, the relevant exponent is $\beta=(1+\rho^2)/\rho^2=(3k-2)/k>2$ for $k\geq 3$, a regime to which Lemma B.3 ($\beta=1$) does not apply.

Lemma B.4 (Generalized Hermite–Gaussian tail recurrence)**.** *For every $\beta>0$, every integer $m\geq 0$, and every $u\in\mathbb{R}$, set*

$$J_m^\beta(u) \;:=\; \int_u^\infty H_m(y)\,e^{-\beta y^2/2}\,\mathrm{d}y\,. \tag{B.5a}$$

Write $\gamma:=2/\beta$ and $\vartheta:=(2-\beta)/\beta$. The base cases are

$$J_0^\beta(u) \;=\; \sqrt{\tfrac{2\pi}{\beta}}\,\bar\Phi(u\sqrt{\beta})\,, \qquad J_1^\beta(u) \;=\; \gamma\,e^{-\beta u^2/2}\,, \tag{B.5b}$$

and for every $m\geq 2$, the family $\{J_m^\beta\}$ satisfies the three-term recurrence

$$J_m^\beta(u) \;=\; \gamma\,H_{m-1}(u)\,e^{-\beta u^2/2} \;+\; 2(m-1)\,\vartheta\,J_{m-2}^\beta(u)\,. \tag{B.5c}$$

Iterating (B.5c) yields the explicit closed form

$$J_m^\beta(u) \;=\; \gamma\,e^{-\beta u^2/2}\sum_{\ell=0}^{\lfloor (m-1)/2\rfloor}(2\vartheta)^\ell\,\frac{(m-1)!!}{(m-2\ell-1)!!}\,H_{m-2\ell-1}(u) \;+\; R_m^\beta(u)\,, \tag{B.5d}$$

where $R_m^\beta(u)=0$ if m is odd, and $R_m^\beta(u)=(2\vartheta)^{m/2}(m-1)!!\,J_0^\beta(u)$ if m is even.

For $\beta=1$, $\gamma=2$ and $\vartheta=1$, so $\gamma(2\vartheta)^\ell=2^{\ell+1}$, and Lemma B.4 reduces exactly to (B.2); the remainder R_m^β specializes to $2^{m/2}(m-1)!!\int_u^\infty e^{-y^2/2}\,\mathrm{d}y$, matching R_m. For $\beta>2$, $\vartheta<0$, and the iterated sum (B.5d) carries alternating signs in ℓ but remains exact pointwise; in subsequent estimates one must not pass to absolute values of individual summands.

Proof of Lemma B.4. • Apply integration by parts with $f(y)=H_{m-1}(y)$ and $g'(y)=2y\,e^{-\beta y^2/2}$, hence $g(y)=-(2/\beta)\,e^{-\beta y^2/2}=-\gamma\,e^{-\beta y^2/2}$. The boundary term at $+\infty$ vanishes by Gaussian decay, and using $H'_{m-1}(y)=2(m-1)\,H_{m-2}(y)$,

$$\int_u^\infty 2y\,H_{m-1}(y)\,e^{-\beta y^2/2}\,\mathrm{d}y \;=\; \gamma\,H_{m-1}(u)\,e^{-\beta u^2/2} \;+\; 2(m-1)\,\gamma\,J_{m-2}^\beta(u)\,.$$

Combining with the Hermite recurrence $H_m(y) = 2y\,H_{m-1}(y) - 2(m-1)\,H_{m-2}(y)$,

$$J_m^\beta(u) \;=\; \int_u^\infty 2y\,H_{m-1}(y)\,e^{-\beta y^2/2}\,\mathrm{d}y \;-\; 2(m-1)\,J_{m-2}^\beta(u)\,,$$

and $2(m-1)\gamma - 2(m-1) = 2(m-1)(\gamma-1) = 2(m-1)\vartheta$, yielding (B.5c).

• For the boundary cases, $J_0^\beta(u) = \int_u^\infty e^{-\beta y^2/2}\,\mathrm{d}y = \sqrt{2\pi/\beta}\,\bar\Phi(u\sqrt{\beta})$ by direct change of variable, and $J_1^\beta(u) = \int_u^\infty 2y\,e^{-\beta y^2/2}\,\mathrm{d}y = (2/\beta)\,e^{-\beta u^2/2} = \gamma\,e^{-\beta u^2/2}$.

• Unrolling (B.5c) ℓ times and using the telescoping product $(m-1)(m-3)\cdots(m-2\ell+1) = (m-1)!!/(m-2\ell-1)!!$,

$$J_m^\beta(u) \;=\; \gamma\,e^{-\beta u^2/2}\sum_{j=0}^{\ell-1}(2\vartheta)^j\,\frac{(m-1)!!}{(m-2j-1)!!}\,H_{m-2j-1}(u) \;+\; (2\vartheta)^\ell\,\frac{(m-1)!!}{(m-2\ell-1)!!}\,J_{m-2\ell}^\beta(u)\,.$$

For m odd, take $\ell = (m-1)/2$: $m-2\ell-1 = 0$, hence $(m-2\ell-1)!! = 0!! = 1$, the residual factor $J_{m-2\ell}^\beta(u) = J_1^\beta(u) = \gamma\,e^{-\beta u^2/2}$, and the residual term coincides with the $j = (m-1)/2$ summand of the main sum (since $H_{m-2\ell-1}(u) = H_0(u) = 1$ and $\gamma e^{-\beta u^2/2} = J_1^\beta(u)$); absorbing it into the main sum gives $R_m^\beta = 0$. For m even, take $\ell = m/2$: $m-2\ell-1 = -1$, $(m-2\ell-1)!! = (-1)!! = 1$, the residual factor is $J_0^\beta(u)$, and $R_m^\beta(u) = (2\vartheta)^{m/2}\,(m-1)!!\,J_0^\beta(u)$.

□

Corollary B.5 (Squared-Hermite Gaussian tail)**.** *For every $\beta > 0$, every integer $j \geq 0$, and every $u \in \mathbb{R}$, the squared-Hermite tail*

$$K_j^\beta(u) \;:=\; \int_u^\infty H_j(y)^2\,e^{-\beta y^2/2}\,\mathrm{d}y \tag{B.6a}$$

admits the exact closed form

$$K_j^\beta(u) \;=\; \sum_{p=0}^{j} 2^p\,p!\,\binom{j}{p}^2\,J_{2j-2p}^\beta(u)\,, \tag{B.6b}$$

where each J_{2j-2p}^β is given in closed form by Lemma B.4.

Proof of Corollary B.5. The classical linearization of the product of two physicist Hermite polynomials reads

$$H_m(y)\,H_n(y) \;=\; \sum_{p=0}^{\min(m,n)} 2^p\,p!\,\binom{m}{p}\binom{n}{p}\,H_{m+n-2p}(y)\,.$$

Specializing to $m = n = j$ yields $H_j(y)^2 = \sum_{p=0}^{j} 2^p\,p!\,\binom{j}{p}^2\,H_{2j-2p}(y)$. Multiplying by $e^{-\beta y^2/2}$, integrating from u to ∞, and exchanging the finite sum with the integral (justified termwise since each $H_m\,e^{-\beta y^2/2}$ is absolutely integrable on $[u,\infty)$), one obtains (B.6b). □

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Statements and Declarations

Competing interests. The authors have no competing interests to declare that are relevant to the content of this article.

Data availability. The scripts that produce the figures and the numerical values reported in this article are available at github.com/ydecastro/TENSOR-KSS. No other data were generated.

Disclosure of AI use

In line with the Leiden Declaration on Artificial Intelligence and Mathematics [11], we disclose the use of AI agents (Claude Code, Anthropic) in the preparation of this paper. These agents were used to audit the statements, the proofs and the numerical tables, through repeated adversarial verification passes recorded in a change-log, and to help with editorial tasks: consistency of notation, bibliography formatting, and typesetting. The authors produced and checked every result and every proof themselves and take sole responsibility for the correctness of the paper and for its citations.

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